Chaos and Undecidability (draft)

E. Asarin

1995

1 Main Thesis

This is a well-known idea:

Thesis 1 *Chaos generates Undecidability.*

Our aim is to refine this by

Thesis 2 *Chaos generates Undecidability via Ability of universal computation.*

2 Mechanisms of undecidability

Let us start by a short discussion of several mechanisms providing undecidability of the reachability (or similar) problem for dynamical systems.

2.1 Amplification of a small ineffectiveness

Something "not very effective" in system description or initial data is amplified by the system's dynamics. The standard undecidability results for computable real number fit in this category as well as the following theorem.

Theorem 1 (Pour-El, Richards) *For the problem*

$$\begin{aligned} \Delta u - \frac{\partial^2 u}{\partial t^2} &= 0 \\ u(x,y,z,0) &= f(x,y,z) \\ \frac{\partial u}{\partial t}(x,y,z,0) &= 0 \end{aligned}$$

there exist such a computable $f(x,y,z)$ that the solution $u(x,y,z,1)$ is not computable.

Let us give a toy example explaining this mechanism. Let $\varphi(m)$ be a recursive integer function with non-recursive range W. Put

$$f(n) = \begin{cases} 2^{-\min\{m: \quad \varphi(m)=n\}} & \text{if } n \in W \\ 0 & \text{if } n \notin W \end{cases}$$

f is a computable function from integers to computable real numbers but if we apply to f an "amplifying" operator sign we obtain sign f i.e. the characteristic function of W which is evidently not computable.

2.2 Non-compact state space

Theorem 2 (Henzinger, Kopke) *Reachability problem for automata with 4 clocks + 1 stopwatch is undecidable.*

I have no comments to this example.

2.3 Zeno's phenomenon

Sometimes undecidability or even higher undecidability can be related with the ability of a dynamical system to perform infinitely many "steps" in finite time. An example of this type follows.

Theorem 3 (Asarin, Maler) *For any arithmetic set of integers there exits an ODE with piecewise constant right-hand side which recognizes this set. this set.*

2.4 Infinitely many regions

. One more toy example: for any configuration of a universal Turing machine take a point in $I\!R^3$ representing this configuration. For each legal transition of the TM (from configuration to configuration) make an arc joining the corresponding points. This construction can be done in an effective way. The halting problem for the original TM can be reduced now to the reachability problem for the infinite (but effective) graph obtained.

There are many real (and senseful) examples generalizing the previous toy example.

This list of mechanisms of undecidability is far from being exhaustive. In the rest of the paper another mechanism is considered.

3 Undecidability of reachability problem for "finite" systems and initial data in compact state space

Let us refine once again our thesis.

Thesis 3 *Any class of "finite" dynamical systems which can simulate Turing machines necessarily contains chaotic systems.*

In a way the inverse also holds

Thesis 4 *If a class of dynamical systems contains a system isomorphic to the baker's transform (i.e. two-way Bernoulli shift) and admits "merging" subsystems then it can simulate Turing machines.*

The techniques of simulation using nothing else then baker's transform and merging subsystems can be found in [AMP].

Now let us consider the precise meaning of thesis 3.

3.1 TM as dynamical systems

Two kinds of Turing machines will be considered

iTM has a state space $\Sigma^{\mathbb{Z}} \times Q$ where Σ stands for its tape alphabet and Q for its control alphabet. Note that infinite tape configurations are allowed. The head of the TM is always at the cell number 0 it is the tape that moves.

fTM. Infinite tape configurations are disallowed for this kind of TM. So the state space of this machine is $\Sigma_f^{\mathbb{Z}} in \times Q$ where $\Sigma_f^{\mathbb{Z}} in$ denotes the subset of $\Sigma^{\mathbb{Z}}$ consisting of all the sequences which contain only finitely many non-blank symbols.

Introduce a metrics on the state-spaces by the formula

$$\rho\left((\alpha, q), (\beta, r)\right) = \begin{cases} 2^{-\min\{|i| : \alpha_i \neq \beta_i\} - 1} & \text{if } q = r \\ 1 & \text{if } q \neq r \end{cases}$$

Provided with such a metrics $\Sigma^{\mathbb{Z}} \times Q$ is homeomorhic to $\#Q$ copies of the Cantor space. The state space of the fTM is a non-complete everywhere dense subspace of the latter.

For both cases denote the transition function of a TM (which acts on its state-space) by T.

Lemma 1 *T is Lipshitz continuous (coefficient 2)*

Proof. There are two cases.

- The distance between two initial configurations $\rho\left((\alpha, q),(\beta, r)\right) \geq 1/2$. Since the distance between any two points cannot be greater than 1 the inequality to prove

$$\rho\left((\alpha, q),(\beta, r)\right) \leq \rho\left(T(\alpha, q), T(\beta, r)\right)$$

  is evident.

- The distance between two initial configurations $\rho\left((\alpha, q),(\beta, r)\right) < 1/2$. In this case the TM is in the same control state (i.e. $q = r$) and the symbol observed is the same (i.e. $\alpha_0 = \beta_0$). This means that the action of the TM should be the same and the only possibility to increase the distance between the states is to move the tape in such a way that $\min\{|i| : \alpha_i \neq \beta_i\}$ decreases. But its decrement can be at most 1, so again

$$\rho\left((\alpha, q),(\beta, r)\right) \leq \rho\left(T(\alpha, q), T(\beta, r)\right)$$

Lemma 2 *There exists an iTM isomorhic to the baker's transform (2-way Bernoulli shift)*

Proof. Consider the machine with 1 control state which always moves its tape to the left without making anything else.

Definition. A dynamical system (M, T) where M is a metric space T its continuous transform is called sensitive to initial values in $x \in M$ if there exists $\gamma > 1, \delta, \tau > 0$ such that for any $\varepsilon > 0$ there exists an y in the ε-neighborhood of x such that for any $k \leq \delta|\ln \varepsilon|$ the inequality $\rho(T^k x, T^k y) \geq \tau\gamma^k \rho(x, y)$ holds.

Lemma 3 *There exists an fTM sensitive to initial conditions in any point of its configuration space with $\gamma = 2$*

Proof. Consider the same machine with 1 control state which always moves its tape to the left without making anything else.

One could suppose that this sensitivity can occur only in such a degenerate kind of TM fact but this is not the case. In a typical case of a TM a difference of 2^{-m} in initial states (that is the tapes being different in m cells from the head) can increase twice at each step (when the head moves towards these different cells) during time m. After that the TM switches to two different control states and the two trajectories evolve in two different ways.

Lemma 4 (Koiran et al.) *There exists a fTM with infinitely many period-k cycles for infinitely many k.*

All these simple lemmata show that Turing machines considered as dynamical systems reveal some aspect of chaos. Further we shall deduce some consequences for dynamical systems capable to model these TM.

3.2 Properties of dynamical systems simulating TM

Introduce a general notion of simulation between dynamical systems and TM.

Definition. Let (M, S) be a dynamical system (i.e. M is a state space, S - its transformation); let (N, T) be an fTM or an iTM (the state space N is as described before). A simulation is a pair (M_1, F) such that

- $M_1 \subset M$;
- M_1 is invariant wrt to S;
- $F : M_1 \to N$;

- $F(M_1) = N$;
- $FS = TF$

We say that a simulation is 1-1, continuous, Lipshitz continuous if so is F (in the last two cases we suppose that M is provided with a topological or metric structure respectively).

This definition made we can obtain some immediate consequences of lemmata concerning TM. (M, S)

Theorem 4 *Let (M, S) be a dynamical system, (N, T) be an fTM or an iTM and (M_1, F) a simulation between them.*

- *if (N, T) is an iTM and the simulation is 1-1 and continuous, then there exists an invariant subset of M (namely M_1) homeomorhic to several copies of the Cantor space;*
- *if (N, T) is an iTM and the simulation is continuous, then there exists an invariant subset of M (namely M_1) which can be continuously mapped onto the Cantor space;*
- *if (N, T) is as in lemma 2 and the simulation is 1-1 then (M_1, S) can be provided with an invariant measure which makes it isomorphic to the 2-way Bernoulli shift;*
- *if (N, T) is as in lemma 2 and the simulation is 1-1 and continuous then (M_1, S) is topologically isomorphic to the 2-way topological Bernoulli shift;*
- *if (N, T) is as in lemma 3 and the simulation is 1-1, Lipshitz continuous and F^{-1} is Lipshitz continuous as well, then (M_1, S) is sensitive to initial conditions;*
- *if (N, T) is as in lemma 4 and the simulation is 1-1 then (M, S) has infinitely many period-k cycles for infinitely many k.*

This theorem is a theoretical explanation of thesis 3.

Now several example are considered.

3.3 Examples of chaotic systems simulating Turing machines

Example 1 (Koiran, Cosnard, Garzon) Any iTM can be simulated by a piecewise linear mapping on the unit square in $\mathbb{R}^2$. A tape configuration $\alpha \in \{0, 1\}^{\mathbb{Z}}$ is represented by a point with coordinates

$$\left(\sum_{i=0}^{\infty} (2\alpha_i + 1) 4^{-i-1}, \sum_{i=1}^{\infty} (2\alpha{-}i + 1) 4^{-i} \right)$$

This point belongs to a "2-dimensional" Cantor set, and M_1 consists of several copies of this set (one for each control state of the machine).

Example 2 (C. Moore) Any iTM can be simulated by a smooth mapping of the unit square in $\mathbb{R}^2$, this mapping can be embedded in a smooth flow in $\mathbb{R}^3$. Encoding and the invariant set M_1 are mainly similar to one of the first example.

Example 3 (Asarin, Maler, Pnueli) Any fTM can be simulated by a flow generated by an ODE with piecewise-constant right-hand side in $\mathbb{R}^3$. There are also invariant Cantor sets. The main block of the construction is a flow simulating baker's transform.

In all these examples several features of deterministic chaos are present.

3.4 Examples of non-chaotic systems unable to simulate Turing machines

Example 1 (Koiran, Cosnard, Garzon) There exists an fTM which cannot be 1-1 simulated by any piecewise-linear mapping on $\mathbb{R}$. This result follows from the following (difficult) lemma.

Lemma 5 (Koiran et al.) *A piecewise-linear mapping on $\mathbb{R}$ cannot have infinitely many period-k cycles for infinitely many k.*

Example 2 (Maler, Pnueli) Reachability between rational points is decidable for ODEs with piecewise-constant right-hand side in $\mathbb{R}^2$. The proof of this result is based on the following lemma.

Lemma 6 (Maler, Pnuely) *The signature of any trajectory of an ODEs with piecewise-constant right-hand side on the plane is ultimately periodic.*

In both examples the proof of impossibility to simulate a TM (or even decidability) uses impossibility of some aspects of chaos for the class of dynamical systems under consideration.