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RATIONAL BASE NUMBER SYSTEMS FOR p-ADIC NUMBERS

Christiane Frougny[1] and Karel Klouda[2]

Abstract. This paper deals with rational base number systems for p-adic numbers. We mainly focus on the system proposed by Akiyama, Frougny, and Sakarovitch in 2008, but we also show that this system is in some sense isomorphic to some other rational base number systems by means of finite transducers. We identify the numbers with finite and eventually periodic representations and we also determine the number of representations of a given p-adic number.

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Introduction

In this paper, we consider four distinct but similar rational base number systems. The starting point of the derivation of all these systems is the classical *Division Algorithm* which computes the representation of positive integers in an integer base $b \geqslant 2$: for nonzero s in $\mathbb{N}$, put $s_0 = s$ and

$$s_i = bs_{i+1} + a_i, \quad a_i \in \{0, 1, \dots, b-1\}, \tag{1}$$

for $i = 0, 1, \dots$. The resulting sequence of *digits* $\cdots a_2 a_1 a_0$ is always finite, meaning that there is some n in $\mathbb{N}$ such that $a_n \neq 0$ and $a_k = 0$ for all $k > n$, and, moreover, it holds that $s = \sum_{i=0}^{n} a_i b^i$. We say that $a_n \cdots a_0$ is the representation of s in base b. The same representation can be obtained also using the well-known *greedy algorithm* only with the difference that it is computed from left to right: the most significant digit a_n first.

If one wants to get a rational base representation by the greedy algorithm, it suffices to replace the integer base b by a rational number $\frac{p}{q}, p > q \geqslant 1$. However, even if the input is a positive integer, the returned representations may be infinite to the right, *i.e.*, the sequence of digits is not eventually zero. In fact, even if the integer base b is replaced by a real $\beta > 1$, the greedy algorithm still works. The resulting representations are called *β-expansions*. The notion of β-expansion was firstly introduced by Rényi in [7] and has been studied since then by many authors (see [3, Chapter 7] for a survey and references). The β-expansions are obtained by the generalization of the greedy algorithm. For the most general setting of the greedy algorithm we have: the input can be any nonnegative real number, any real number greater than one can be taken as a base, and the β-expansions are, in general, infinite to the right.

Keywords and phrases: rational base number systems, p-adic numbers

[1] LIAFA, CNRS UMR 7089, Case 7014, 75205 Paris Cedex 13, and Université Paris 8; e-mail: `Christiane.Frougny@liafa.jussieu.fr`
[2] Faculty of Information Technology, Kolejní 550/2, 160 00 Prague; e-mail: `karel.klouda@fit.cvut.cz`

The goal of this paper is to study some generalizations of the division algorithm in the case where the input is not a positive integer, and in the case where the base is not a positive integer.

As for the possible inputs, looking at the key step of the Division Algorithm (1), an irrational number can be hardly an input. As we will see below, the algorithm can be modified so that the input can be rational numbers.

Regarding the possible bases, again an irrational base is not acceptable. In order to get a rational base number system, we have to modify (1): let $p > q \geqslant 1$ be co-prime integers, if we replace (1) by

$$qs_i = ps_{i+1} + a_i, \quad \text{or by} \quad qs_i = ps_{i+1} + qa_i, \tag{2}$$

where $a_i \in \{0, 1, \ldots, p-1\}$, we get the rational base number systems we are going to study. It is easy to check that for any positive integer $s = s_0$ we get

$$s = \sum_{i=0}^{n_1} \frac{a_i}{q} \left(\frac{p}{q}\right)^i \quad \text{or} \quad s = \sum_{i=0}^{n_2} a_i \left(\frac{p}{q}\right)^i, \quad \text{for some } n_1, n_2 \in \mathbb{N}, \tag{3}$$

respectively. If we further replace p by $-p$, we get two negative rational base systems. Again, all the respective algorithms admit any rational number as an input (see Algorithm 2.1).

We have said that even negative numbers can be an input of the division algorithm: let $b = 2$ and $s = -1$ in (1), then the output is the left-infinite sequence $\cdots a_2 a_1 a_0 = \cdots 111$. In order for this sequence to be the representation of -1, we would have to have $-1 = \sum_{i=0}^{\infty} 2^i$. This is of course not true with respect to the classical absolute value, therefore we have to move to another field. In some sense, the only candidate is the field of 2-adic numbers $\mathbb{Q}_2$. The fields of r-adic numbers $\mathbb{Q}_r$, for r a prime number, will be described in the sequel.

Now, we can specify what this paper deals with: we will study four rational base number systems enabling to represent r-adic numbers in the form

$$\sum_{i \geqslant k_0} a_i \left(\frac{p}{q}\right)^i, \quad \sum_{i \geqslant k_0} a_i \left(-\frac{p}{q}\right)^i, \quad \sum_{i \geqslant k_0} \frac{a_i}{q} \left(\frac{p}{q}\right)^i, \quad \sum_{i \geqslant k_0} \frac{a_i}{q} \left(-\frac{p}{q}\right)^i, \tag{4}$$

where $p > q \geqslant 1$ are co-prime integers, a_i belongs to $\{0, 1, \ldots, p-1\}$ and k_0 is in $\mathbb{Z}$. Our strategy is to study one of them and then to show that they all share most of their properties. More precisely, we will study the third one, called the *AFS number system*, since this system has already been considered by Akiyama, Frougny, and Sakarovitch in [1]. We first study the representations of the negative integers, and show that they are representable by a tree, similar to the tree of the representations of nonnegative integers of [1]. We characterize the case when there is a natural isomorphism of the trees, Proposition 2.13.

In Proposition 2.17 we give a combinatorial description of the numbers having a finite expansion.

Theorem 2.20 provides an answer to the question of uniqueness of a representation and also characterizes all representations of x in $\mathbb{Q}_r$, r a prime factor of p, which converge to x with respect to $|\,.\,|_r$.

Then we characterize the numbers with eventually periodic representations: more precisely, we show that if x belongs to $\mathbb{Q}_r$, r a prime factor of p, then the $\frac{1}{q}\frac{p}{q}$-representation of x, given by Algorithm GMD, is eventually periodic if, and only if, x is in $\mathbb{Q}$ and the $\frac{1}{q}\frac{p}{q}$-representation is equals to the $\frac{1}{q}\frac{p}{q}$-expansion of x given by Algorithm MD, Theorem 2.26.

Finally we show that, for $q > 1$, there exist finite sequential transducers converting one representation from (4) to each other one, Theorem 3.1.

1. Preliminaries

1.1. p-adic numbers

Within this section, p is a prime number. Detailed introduction to the theory of p-adic numbers can be found in many books, see, *e.g.*, [5]. Here we shortly recall the definition and some basic properties we are going to need later on. First, define the *p-adic valuation* $v_p : \mathbb{Z} \setminus \{0\} \to \mathbb{R}$ by $n = p^{v_p(n)} n'$ with $p \nmid n'$. This is extended to rationals by $v_p(\frac{a}{b}) = v_p(a) - v_p(b)$ for any nonzero $a, b \in \mathbb{Z}$. Having the valuation, we define the *p-adic absolute value* of x in $\mathbb{Q}$ as $|x|_p = 0$ if $x = 0$, and $|x|_p = p^{-v_p(x)}$ otherwise. Due to the celebrated Ostrowski's theorem from 1918, the p-adic absolute values and the classical one are the only non-trivial absolute values definable on $\mathbb{Q}$ since the theorem says any absolute value is equivalent to one of these.

The crucial difference between the classical and the p-adic absolute value is that the p-adic one is *ultrametric*, *i.e.*, for all $x, y \in \mathbb{Q}_p$ it holds $|x+y|_p \leqslant \max\{|x|_p, |y|_p\}$. Moreover, even for non-rational p-adic numbers the absolute value (and valuation) still takes only countably many values; more precisely, there exists $i \in \mathbb{Z}$ such that $|x|_p = p^{v_p(x)} = p^i$ for all $x \in \mathbb{Q}_p$.

In the same way as the set $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the classical absolute value, the sets $\mathbb{Q}_p$ of p-adic numbers are the completions of $\mathbb{Q}$ with respect to $| \ |_p$. It is known that any x in $\mathbb{Q}_p$ has a unique *standard representation* in base p:

$$x = \sum_{k \geqslant -k_0} a_k p^k, \quad \text{with } a_k \in \{0, 1, \ldots, p-1\} \text{ and } k_0 = v_p(x).$$

This standard representation is finite if, and only if, x is in $\mathbb{N}$, eventually periodic for x in $\mathbb{Q}$ and aperiodic otherwise. The existence and uniqueness of the standard representation implies the following simple but essential lemma; $\mathbb{Z}_p$ is the set of p-adic integers, *i.e.*, numbers $x \in \mathbb{Q}_p$ with $|x|_p \leqslant 1$.

Lemma 1.1. *Let $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$. Then there exists a unique $\alpha_n \in \{0, 1, \ldots, p^n - 1\}$ such that $|x - \alpha_n|_p \leqslant p^{-n}$.*

1.2. Combinatorics on words

Any finite nonempty set $\mathcal{A}$ is called *alphabet*. In particular, we put $\mathcal{A}_k = \{0, 1, \ldots, k-1\}$ for any $k \in \mathbb{N}$. Any finite string $w = a_0 a_1 \cdots a_n$, $a_i \in \mathcal{A}$, is a *finite word* over $\mathcal{A}$ of length $|w| = n+1$. The set of all finite words over $\mathcal{A}$ including the *empty word* ε is denoted by $\mathcal{A}^*$. A *right-infinite word* over $\mathcal{A}$ is a sequence $\mathbf{a} = a_0 a_1 \cdots$ with $a_i \in \mathcal{A}$; $\mathcal{A}^{\mathbb{N}}$ is the set of all such words over $\mathcal{A}$. If $\mathbf{a} = uwww \cdots = uw^\omega$ for some u and w in $\mathcal{A}^*$, then $\mathbf{a}$ is said to be *eventually periodic*. If $u = \varepsilon$, $\mathbf{a}$ is *purely periodic*. Left-infinite words and the set ${}^{\mathbb{N}}\!\mathcal{A}$ are defined in the same manner. If $\mathbf{a} = \cdots wwwu = {}^{\omega}wu$ for some $w, u \in \mathcal{A}^*$, then $\mathbf{a}$ is *eventually periodic* to the left.

If w in $\mathcal{A}^*$ is equal to zu for some z and u in $\mathcal{A}^*$, then z is a *prefix* and u a *suffix* of w. A *language L* over $\mathcal{A}$ is any subset of $\mathcal{A}^*$. If any prefix of any $w \in L$ belongs also to L, L is a *prefix-closed language*.

2. AFS number system

In [1] the AFS system is proposed as a new method to represent the nonnegative integers in the form of the third series from (4), where $p > q \geqslant 1$ are co-prime integers and digits a_i from the alphabet $\mathcal{A}_p$. It is proved there that such a finite representation is unique and that the language of all such representations is prefix-closed. In fact, it holds that if w in $\mathcal{A}_p^*$ is a representation of an integer, then there exists at least one a in $\mathcal{A}_p$ such that wa represents an integer as well. So, if $w = a_n a_{n-1} \cdots a_1 a_0$, we can study $\sum_{k=0}^{n} \frac{a_k}{q} \left(\frac{p}{q}\right)^{-k}$ and get a representation of a rational number. As we have said, w can always be extended by at least one letter and remains a representation of an integer. Doing this extension repetitively, n tends to infinity and we can get even irrational numbers. Such infinite representations are then studied in [1] and they turn out to be very interesting and to relate to old and difficult problems of Number Theory; namely, Mahler's problem [4] and the Josephus problem [6], [8]. In this paper we will take a different approach; we will study also infinite series but containing an infinite number of *positive* powers of $\frac{p}{q}$.

2.1. Modified division algorithm

In what follows, we assume that $p > q \geqslant 1$ are co-prime positive integers (we do *not* assume that p is a prime number!). As explained above, the representations in the AFS system can be obtained by the Division Algorithm if the key step (1) is modified. The result is the following algorithm, called the *Modified Division (MD) algorithm*. It is stated in the most general form so that its input can be any rational number $x = \frac{s}{t}$ with s and t being the lowest terms.

Algorithm 2.1 (MD algorithm)**.** *Let $x = \frac{s}{t}$, s being an integer and t a positive integer.*

(i) *If $s = 0$, return the empty word $\mathbf{a} = \varepsilon$.*

(ii) *If t is co-prime to p, put $s_0 = s$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by*

$$\frac{qs_i}{t} = \frac{ps_{i+1}}{t} + a_i. \tag{5}$$

Return $\mathbf{a} = \cdots a_2 a_1 a_0$.

(iii) *If t is not mutually prime with p, multiply $\frac{s}{t}$ by $\frac{p}{q}$ until $x\left(\frac{p}{q}\right)^{\ell}$ is of the form $\frac{s'}{t'}$, where t' is co-prime to p. Then apply the algorithm from (ii) returning $\mathbf{a}' = \cdots a'_2 a'_1 a'_0$. Return $\mathbf{a} = \cdots a_1 a_0 \,\mathbf{.}\, a_{-1} \cdots a_{-\ell} = \cdots a'_{\ell+1} a'_{\ell} \,\mathbf{.}\, a'_{\ell-1} \cdots a'_0$.*

Definition 2.2. Let x be in $\mathbb{Q}$. The word $\mathbf{a}$ returned by the previous algorithm for x is said to be the $\frac{1}{q}\frac{p}{q}$*-expansion* of x and denoted by $\langle x \rangle_{\frac{1}{q}\frac{p}{q}}$.

We often omit the radix point if its position is clear.

Lemma 2.3. *Let $x = \frac{s}{t}$, where $s \neq 0$ and $t > 0$ is co-prime to p. Then for the sequence $(s_i)_{i \geqslant 1}$ from the MD algorithm we have:*

(i) *If $s > 0$ and $t = 1$, i.e., $x \in \mathbb{N}$, $(s_i)_{i\geqslant 1}$ is eventually zero.*

(ii) *If $s > 0$ and $t > 1$, $(s_i)_{i\geqslant 1}$ is either eventually zero or eventually negative.*

(iii) *If $s < 0$, $(s_i)_{i\geqslant 1}$ is negative.*

(iv) *For all $i \in \mathbb{N}$, if $s_i < -\frac{p-1}{p-q}t$, then $s_i < s_{i+1}$.*

(v) *For all $i \in \mathbb{N}$, if $-\frac{p-1}{p-q}t \leqslant s_i < 0$, then $-\frac{p-1}{p-q}t \leqslant s_{i+1} < 0$.*

(vi) *$(s_i)_{i\geqslant 1}$ is always bounded and eventually periodic.*

(vii) *$(s_i)_{i\geqslant 1}$ is eventually zero (resp. eventually periodic) if, and only if, $\mathbf{a}$ is eventually zero (resp. eventually periodic).*

Proof. Items (i), (ii) and (iii) follow from the trivial fact that if s_i is positive, then $s_{i+1} < s_i$, and that if s_i is negative, then s_{i+1} is also negative.

If $s_i < -\frac{p-1}{p-q}t$, we must have $s_i < \frac{qs_i-(p-1)t}{p}$. And since

$$\frac{qs_i - (p-1)t}{p} \leqslant s_{i+1} \leqslant \frac{q}{p}s_i$$

for all $i \in \mathbb{N}$, item (iv) follows.

Similarly one can prove (v). Let $-\frac{p-1}{p-q}t \leqslant s_i < 0$, then by (iii) we get $s_{i+1} < 0$ and

$$s_{i+1} \geqslant \frac{qs_i - (p-1)t}{p} \geqslant \frac{q\left(-t\frac{p-1}{p-q}\right) - (p-1)t}{p} = -t\frac{p-1}{p-q} .$$

Item (vi) is a direct consequence of (ii) – (iv) and of the fact that the value of s_{i+1} (and also of a_i) is completely determined by the value of s_i. The same fact implies (vii). □

It follows from the lemma that the interval $[-t\frac{p-1}{p-q}, 0]$ is a sort of attractor for the sequence $(s_i)_{i\geqslant 1}$ and this gives us the following bound for the length of the period of $\frac{1}{q}\frac{p}{q}$-expansions.

Corollary 2.4. *Let $x = \frac{s}{t}$ be in $\mathbb{Q}$. Then the period of $\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$ is less than $\left\lfloor\frac{p-1}{p-q}\right\rfloor t$.*

Lemma 2.5. *Let $x = \frac{s}{t}$ be in $\mathbb{Q}$ such that its $\frac{1}{q}\frac{p}{q}$-expansion $\langle x\rangle_{\frac{1}{q}\frac{p}{q}} = \cdots a_{-\ell+1}a_{-\ell}$, $\ell \in \mathbb{N}$, is not finite (i.e., it is not eventually zero). Then*

$$\sum_{k=-\ell}^{\infty} \frac{a_k}{q}\left(\frac{p}{q}\right)^k$$

converges to x with respect to the r-adic absolute value $|\ |_r$ if, and only if, r is a prime factor of p. Moreover, if i is the multiplicity[1] *of r in p, then for all $n \geqslant -\ell$ we have*

$$\left| x - \sum_{k=-\ell}^{n} \frac{a_k}{q}\left(\frac{p}{q}\right)^k \right|_r \leqslant r^{-i(n+1)}. \tag{6}$$

Proof. W.l.o.g., assume that t is co-prime to p. Then it follows from (5)

$$\frac{s}{t} = \frac{p}{q}\frac{s_1}{t} + \frac{a_0}{q} = \cdots = \left(\frac{p}{q}\right)^{n+1} \frac{s_{n+1}}{t} + \sum_{k=0}^{n} \frac{a_k}{q}\left(\frac{p}{q}\right)^k .$$

Since $(s_i)_{i\geqslant 1}$ is a sequence of integers, we have $0 < |s_i|_r \leqslant 1$ for all i. Hence,

$$\left| x - \sum_{k=-\ell}^{n} \frac{a_k}{q}\left(\frac{p}{q}\right)^k \right|_r = \left|\left(\frac{p}{q}\right)^{n+1}\right|_r \left|\frac{s_{n+1}}{t}\right|_r \leqslant \left|\left(\frac{p}{q}\right)^{n+1}\right|_r \left|\frac{1}{t}\right|_r .$$

Obviously, this sequence tends to zero if, and only if, r is a prime factor of p. In such a case, we have $\left|\left(\frac{p}{q}\right)^{n+1}\right|_r \left|\frac{1}{t}\right|_r = r^{-i(n+1)}$. □

Of course, Inequality (6) holds even without assuming that $\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$ is not eventually zero. Some examples of $\frac{1}{q}\frac{p}{q}$-expansions are stated in Table 1.

2.2. $\frac{1}{q}\frac{p}{q}$-expansions of the negative integers

The case of $\frac{1}{q}\frac{p}{q}$-expansions of the positive integers has already been studied in [1]. In the present subsection, we will study the case of the negative integers.

Definition 2.6. Let $\cdots a_{-\ell+1}a_{-\ell}$, $\ell \in \mathbb{N}$ be an eventually periodic word over $\mathcal{A}_p$. The *evaluation map* π is defined by: $\pi(\cdots a_{-\ell+1}a_{-\ell}) = x$ if, and only if, $\langle x\rangle_{\frac{1}{q}\frac{p}{q}} = \cdots a_{-\ell+1}a_{-\ell}$.

[1] This means that i is the greatest integer such that r^i divides p.

x	$\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$	$(s_i)_{i\geqslant 0}$	abs. values
$p=3, q=2$			
5	2101	5, 3, 2, 1, 0, 0, ...	all
-5	${}^{\omega}2012$	-5, -3, -2, -2, -2, ...	$\mid\ \mid_3$
11/4	201	11,6, 4, 0, 0, ...	all
11/8	${}^{\omega}1222$	11, 2, -4, -8, -8, -8, ...	$\mid\ \mid_3$
11/5	${}^{\omega}(02)2112$	11, 4, 1, -1, -4, -6, -4, -6, ...	$\mid\ \mid_3$
$p=30, q=11$			
5	11 25	5, 1, 0, 0, ...	all
-5	${}^{\omega}19\ 8\ 5$	-5, -2, -1, -1,...	$\mid\ \mid_2, \mid\ \mid_3, \mid\ \mid_5$
11/7	${}^{\omega}(12\ 21\ 5)\ 23\ 13$	11, 1, -5, -3, -6, -5, ...	$\mid\ \mid_2, \mid\ \mid_3, \mid\ \mid_5$

TABLE 1. Examples of $\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$. The last column contains the absolute values for which the $\frac{1}{q}\frac{p}{q}$-expansion from the second column converges to x (in terms of Lemma 2.5).

Lemma 2.7. *Let $\cdots a_2a_1a_0$ be eventually periodic.*

(i) *If $\pi(\cdots a_2a_1a_0)$ is in $\mathbb{Z}$, then $\pi(\cdots a_3a_2a_1)$ belongs to $\mathbb{Z}$.*
(ii) *If $x=\pi(\cdots a_2a_1a_0)$ is a negative integer, then there exists $a\in\mathcal{A}_p$ such that $\pi(\cdots a_2a_1a_0a)$ is also a negative integer. Moreover,*

$$\min\{\pi(\cdots a_2a_1a_0a)\mid \pi(\cdots a_2a_1a_0a)\in\mathbb{Z}, a\in\mathcal{A}_p\} = \left\lceil x\frac{p}{q}\right\rceil \tag{7}$$

$$\max\{\pi(\cdots a_2a_1a_0a)\mid \pi(\cdots a_2a_1a_0a)\in\mathbb{Z}, a\in\mathcal{A}_p\} = \left\lfloor \frac{1}{q}(px+p-1)\right\rfloor \tag{8}$$

Proof. Let us assume that an integer x has expansion $\langle x\rangle_{\frac{1}{q}\frac{p}{q}} = \cdots a_3a_2a_1a_0$ and let s_1 be the second element of the corresponding sequence $(s_i)_{i\geqslant 0}$ from the MD algorithm. Then, clearly, $\langle s_1\rangle_{\frac{1}{q}\frac{p}{q}} = \cdots a_3a_2a_1$.

It remains to prove (ii). In other words, we want to prove that there exists an integer s such that s_1 from the MD algorithm for $s_0=s$ is equal to x, *i.e.*, $qs=px+a$ for some $a\in\mathcal{A}_p$. It is equivalent to the condition

$$\frac{1}{q}(px+a)\in\mathbb{Z}. \tag{9}$$

Clearly, this condition is satisfied at least for one a, the rest of the statement (ii) follows from the fact that $0\leqslant a\leqslant p-1$. □

In words, the set of $\frac{1}{q}\frac{p}{q}$-expansions of all negative integers is prefix-closed and all its elements are extendable to the right. Moreover, the $\frac{1}{q}\frac{p}{q}$-expansion of a negative integer is eventually periodic with period 1:

Proposition 2.8. *Let k be a positive integer. Denote $B=\left\lfloor\frac{p-1}{p-q}\right\rfloor$, then:*

(i) *if $k\leqslant B$, then $\langle -k\rangle_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}b$ with $b=k(p-q)$,*
(ii) *otherwise, $\langle -k\rangle_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}bw$ with $w\in\mathcal{A}_p^+$ and $b=B(p-q)$.*

Proof. Let $k\leqslant B$. Then, for $s_0=-k$, we have $-qk=qs_0=s_1p+a_0$; this equation is satisfied (only) for $s_1=-k$ and $a=k(p-q)\leqslant p-1$ and the proof of (i) follows.

If $k > B$ then $-k < -\frac{p-1}{p-q}$. We know, due to Lemma 2.3 (iv), that $(s_i)_{i\geqslant 0}$ is eventually greater than or equal to $-B$. Hence, $\langle -k\rangle_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}bw$, where $\pi({}^{\omega}b)$ is a negative integer $-k_1$, $k_1 \leqslant B$. We will prove that k_1 must be equal to B and so, due to (i), $b = B(p-q)$.

We show that if $k_1 < B$ and $b_1 = k_1(p-q)$, then the only $a \in \mathcal{A}_p$ such that $\pi({}^{\omega}b_1a)$ is an integer is again $a = b_1$. Let $k_1 < B$, then $\langle -k_1\rangle_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}b_1$ with $b_1 = k_1(p-q)$. Assume that $k_0 = \pi({}^{\omega}b_1a) \in \mathbb{Z}$. We must have $-k_0q = -k_1p + a$. This is satisfied for $a = k_1(p-q)$ and $k_0 = k_1$. Let us suppose that the same equation is satisfied also for different a' and k_0'. Clearly, a' must be equal to $k_1(p-q) + \ell q$ for some nonzero ℓ and, at least for one of $\ell = 1$ or $\ell = -1$, it must be true that $k_1(p-q) + \ell q \in \mathcal{A}_p$. Then $k_0' = k_1 - 1$ or $k_0' = k_1 + 1$, respectively. It implies that, for some $0 \leqslant k_0' \leqslant B$, we have two different $\frac{1}{q}\frac{p}{q}$-expansions: one is given by (i) (or is equal to ε for $k_0' = 0$) and the second one is ${}^{\omega}b_1a'$, a contradiction. □

2.2.1. *Trees $T_{\frac{1}{q}\frac{p}{q}}$ and $\overline{T}_{\frac{1}{q}\frac{p}{q}}$*

The language of $\frac{1}{q}\frac{p}{q}$-expansions of all positive integers is studied in [1]. It is proved there, among other properties of this language, that it is prefix-closed and extendable to the right. Thus, it is quite natural to represent the language as a tree with infinite branches. We first recall the results for the case of positive integers and then propose their analogues for the negative case.

Lemma 2.9. *Define the language $L_{\frac{1}{q}\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid w$ is the $\frac{1}{q}\frac{p}{q}$-expansion of some $s \in \mathbb{N}\}$. The language $L_{\frac{1}{q}\frac{p}{q}}$ is prefix-closed, extendable to the right, and not context-free (if $q \neq 1$).*

The proof is a direct consequence of the Pumping lemma and can be found in [1].

Definition 2.10. The tree $T_{\frac{1}{q}\frac{p}{q}}$ has the nonnegative integers as nodes and the directed edges are labeled by letters from $\mathcal{A}_p$. Furthermore:

- (i) 0 is the root of the tree,
- (ii) there is an edge from node n_1 to node n_2 with label a if $n_2 = (pn_1 + a)/q$.

Tree $T_{\frac{1}{q}\frac{p}{q}}$ for $p = 3, q = 2$, is depicted in Figure 1.

It is reasonable to ask which nonnegative integer x is the least one with $\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$ of length n. Denote such an integer by G_n: surely $G_0 = 0$ and $G_1 = 1$. The children in the tree $T_{\frac{1}{q}\frac{p}{q}}$ of node n are given by Condition (9), obviously, the least such integer is $\lceil \frac{p}{q}n \rceil$ (*c.f.* Lemma 2.7).

Lemma 2.11. *The least nonnegative integer with $\frac{1}{q}\frac{p}{q}$-expansion of length $n \in \mathbb{N}$ is G_n, where $G_0 = 0$,*

$$G_1 = 1, \quad G_{n+1} = \left\lceil \frac{p}{q}G_n \right\rceil.$$

We now propose equivalent objects for the negative integers. The language now reads $\overline{L}_{\frac{1}{q}\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid {}^{\omega}bw =< s >_{\frac{1}{q}\frac{p}{q}}, s \leqslant -B, \text{first letter of } w \neq b\}$. Clearly, the letter b is equal to $B(p-q)$ with B from Proposition 2.8. Using the same techniques, on can prove the same results as the ones of Lemma 2.9 for the language $\overline{L}_{\frac{1}{q}\frac{p}{q}}$.

Since both languages have the same properties, $\overline{L}_{\frac{1}{q}\frac{p}{q}}$ can be also represented by a tree $\overline{T}_{\frac{1}{q}\frac{p}{q}}$. The nodes are the negative integers, the root is equal to $-B$, and there is an edge from node n_1 to node n_2 with label a in $\mathcal{A}_p$ if $n_2 = (pn_1 + a)/q$.

Tree $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ for $p = 3, q = 2$ is depicted in Figure 1, too.

Again, we can ask which integer $\leqslant -B$ is the least one having the $\frac{1}{q}\frac{p}{q}$-expansion ${}^{\omega}bw$ with $w \in \overline{L}_{\frac{1}{q}\frac{p}{q}}$ of length n. Using the same reasoning as in the case of positive integers, we get:

Lemma 2.12. *The least negative integer with* $\frac{1}{q}\frac{p}{q}$*-expansion* ${}^{\omega}bw$ *with* $b = B(p-q)$ *and* $w \in \overline{L}_{\frac{1}{q}\frac{p}{q}}$ *of length* $n \in \mathbb{N}$ *is* $\overline{G}_n$*, where* $\overline{G}_0 = -B, \quad \overline{G}_{n+1} = \left\lceil \frac{p}{q}\overline{G}_n \right\rceil$.

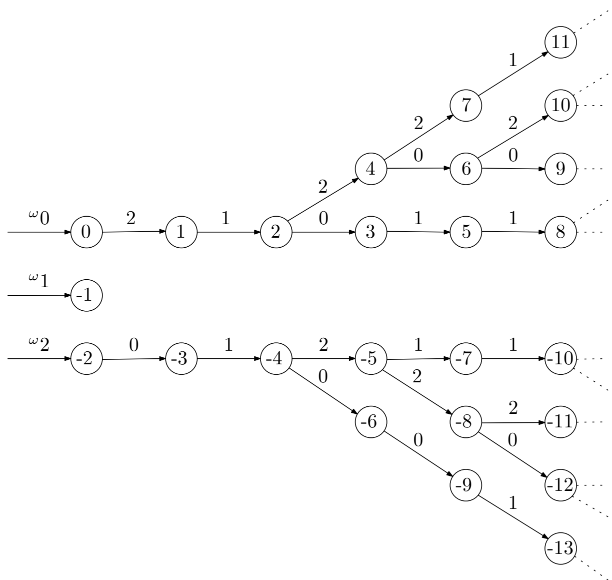

FIGURE 1. The graph containing $\frac{1}{q}\frac{p}{q}$-expansions of all integers for $p = 3, q = 2$: the trees $T_{\frac{1}{q}\frac{p}{q}}$ and $\overline{T}_{\frac{1}{q}\frac{p}{q}}$.

Looking at the trees for various values of p and q, one can notice that sometimes they are isomorphic and sometimes not. For instance, $T_{\frac{1}{q}\frac{p}{q}}$ and $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ are isomorphic for $p = 3$ and $q = 2$ but not for $p = 8$ and $q = 5$, see Figure 2.

Proposition 2.13. *The mapping which maps node* k *of the tree* $T_{\frac{1}{q}\frac{p}{q}}$ *to node* $-B-k$ *of* $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ *is an isomorphism if, and only if,* $\frac{p-1}{p-q}$ *is an integer.*

Proof. Clearly, the mapping is an isomorphism if, and only if, the nodes k and $-B-k$ have the same number of children for all $k \in \mathbb{N}$. Denote the number of children of a node k by $m(k)$, then (see (9)) we have for all $k \in \mathbb{N}$

$$\begin{aligned} m(k) &= \#\{a \in \mathcal{A}_p \mid pk + a \equiv 0 \pmod q\}, \\ m(-B-k) &= \#\{b \in \mathcal{A}_p \mid -p(B+k) + b \equiv 0 \pmod q\} = \\ &= \#\{b \in \mathcal{A}_p \mid pk + (p-q)B - b \equiv 0 \pmod q\}. \end{aligned}$$

The case $q = 1$ is trivial, so assume that $q > 1$. Since p and q are co-prime, the sequence $(pk)_{k \geqslant 0}$ "visits" all residue classes (mod q). Therefore, $m(k) = m(-B-k)$ for all k if, and only if, the set $\{(p-q)B - b \mid b \in \mathcal{A}_p\}$ has the same number of elements in each residue class (mod q) as $\mathcal{A}_p$. This is equivalent to $\{(p-q)B - b \mid b \in \mathcal{A}_p\} = \{\ell q, \ell q + 1, \ldots, \ell q + p - 1\}$ for some $\ell \in \mathbb{Z}$. But since $p-q \leqslant (p-q)B \leqslant p-1$, the only admissible case is that $\ell = 0$ and, consequently, $B = \left\lfloor \frac{p-1}{p-q} \right\rfloor = \frac{p-1}{p-q}$. $\square$

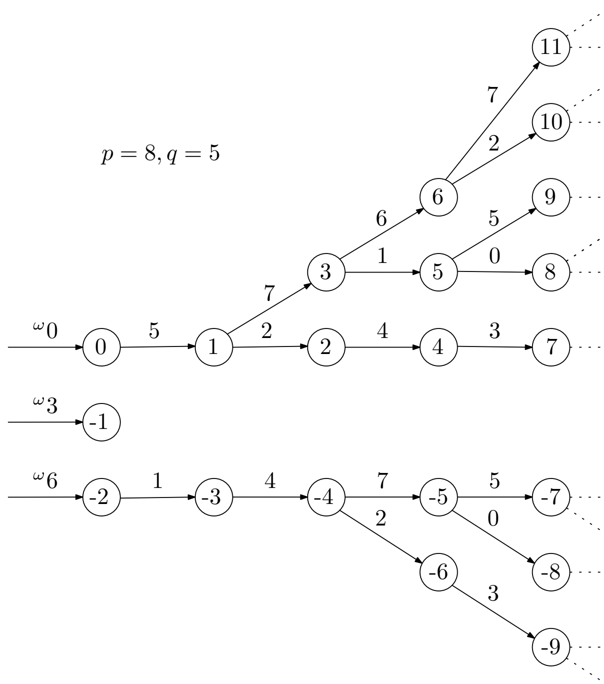

FIGURE 2. The graph containing $\frac{1}{q}\frac{p}{q}$-expansions of all integers for $p = 8, q = 5$: the trees $T_{\frac{1}{q}\frac{p}{q}}$ and $\overline{T}_{\frac{1}{q}\frac{p}{q}}$.

2.3. FINITE $\frac{1}{q}\frac{p}{q}$-EXPANSIONS

If x has a finite $\frac{1}{q}\frac{p}{q}$-expansion of length $m+1$, *i.e.*, $x = \sum_{k=0}^{m} \frac{a_k}{q}\left(\frac{p}{q}\right)^k$, then it is equal to $\frac{s}{q^{m+1}}$ for some $s \geqslant 1$. But not all numbers of this form have a finite $\frac{1}{q}\frac{p}{q}$-expansion, *e.g.*, $x = 11/8 = 11/2^3$ has an eventually periodic representation $^\omega 1222$ for $p = 3$ and $q = 2$, see Table 1. In order to better understand this, we introduce an alternative algorithm computing the $\frac{1}{q}\frac{p}{q}$-expansion of numbers of this form.

Algorithm 2.14. *Let $x = \frac{s}{q^m}$ with s and m positive integers. Put $h_0 = s$. Define h_{i+1} and b_i in $\mathcal{A}_p$ as follows. For $i = 0, 1, \ldots, m-1$ let*

$$\frac{h_i}{q^{m-(i+1)}} = p\frac{h_{i+1}}{q^{m-(i+1)}} + b_i.$$

For $i \geqslant m$ let $qh_i = ph_{i+1} + b_i$.
Return $\mathbf{b} = \cdots b_2b_1b_0$.

It turns out that $\mathbf{b} = \langle x \rangle_{\frac{1}{q}\frac{p}{q}}$, as proved in the following result.

Lemma 2.15. *Let $x = \frac{s}{q^m}$ with s and m positive integers. Let $\mathbf{b} = \cdots b_2b_1b_0$ be the word returned by Algorithm 2.14 and $(h_i)_{i\geqslant 1}$ the respective sequence; similarly, let $\langle x \rangle_{\frac{1}{q}\frac{p}{q}} = \cdots a_2a_1a_0$ and let $(s_i)_{i\geqslant 1}$ be the sequence from the MD algorithm. Then $\langle x \rangle_{\frac{1}{q}\frac{p}{q}} = \mathbf{b}$ and*

$$s_i = \begin{cases} h_i q^i & i = 0, 1, \ldots, m-1, \\ h_i q^m & i = m, m+1, \ldots \ . \end{cases}$$

Proof. For $i = 0$, we have $qs_0 = ps_1 + a_0q^m$ and $h_0 = ph_1 + b_0q^{m-1}$. It follows that $a_0 = b_0$ and $s_1 = qh_1$. We carry on by induction on $i = 1, 2, \dots, m-1$. Assuming that $s_i = h_iq^i$, the equations $qs_i = ps_{i+1} + a_0q^m$ and $h_i = ph_{i+1} + b_iq^{m-(i+1)}$ again imply that $b_i = a_i$ and $s_{i+1} = h_{i+1}$; it suffices to multiply the latter one by q^{i+1} to make it clear.

The proof continues in an analogous way even for i greater than $m-1$. □

It is easy to see that (as in the case of the MD algorithm) if $\cdots b_2b_1b_0$ is the $\frac{1}{q}\frac{p}{q}$-expansion returned by the alternative algorithm for $\frac{h_0}{q^m}$, then $\cdots b_3b_2b_1$ is the $\frac{1}{q}\frac{p}{q}$-expansion of $\frac{h_1}{q^{m-1}}$. Consequently, $\cdots b_{m+2}b_{m+1}b_m$ is the $\frac{1}{q}\frac{p}{q}$-expansion of the integer h_m. We already know that the $\frac{1}{q}\frac{p}{q}$-expansion of an integer is finite if, and only if, the integer is nonnegative, so this implies:

Corollary 2.16. *Let $x = \frac{s}{q^m}$ with s and m positive integers, and let $(h_i)_{i\geqslant 1}$ be the sequence constructed for x in Algorithm 2.14. Then $\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$ is finite if, and only if, h_m is a nonnegative integer.*

Having this knowledge, we are now able to describe all numbers of the form of $\frac{s}{q^m}$ whose $\frac{1}{q}\frac{p}{q}$-expansion is infinite.

Proposition 2.17. *Let $q > 1$. Define for all positive integers m the set*

$$INF(m) = \left\{ i \,\middle|\, i > 0, \left\langle \frac{i}{q^m} \right\rangle_{\frac{1}{q}\frac{p}{q}} \text{ is infinite} \right\}.$$

Then $INF(1) = \emptyset$ and $INF(m) = A(m) \cup B(m)$, $m = 2, 3, \dots,$ where

$$A(m) = \left\{-kp + aq^{m-1} \mid k > 1, a \in \mathcal{A}_p\right\} \cap \mathbb{N}, \quad \text{and} \quad B(m) = \left\{pk + aq^{m-1} \mid k \in INF(m-1), a \in \mathcal{A}_p\right\}.$$

Proof. For $m = 1$ and $\frac{s}{q}, s > 0$, we get in Algorithm 2.14 $s = h_0 = ph_1 + b_0$, it implies $h_1 \geqslant 0$ and so, indeed, INF(1) is empty.

Now, consider $\frac{s}{q^m}$ for $m > 1$. The proof follows from the fact that $s \in \mathrm{INF}(m)$ if, and only if, either $h_1 < 0$ or $h_1 \in \mathrm{INF}(m-1)$. Indeed, if $s \in A(m-1)$, *i.e.*, $s = -kp + aq^{m-1}$ for some $k > 1$ and $a \in \mathcal{A}_p$, then we get in Algorithm 2.14 $s = h_0 = -kp + aq^{m-1} = ph_1 + b_0q^{m-1}$. Since h_1 and b_0 are uniquely given, $h_1 = -k < 0$. Analogously, $s = h_0 = pk + aq^{m-1}$ for some $k \in \mathrm{INF}(m-1)$ and $a \in \mathcal{A}_p$ implies that $h_1 = k \in \mathrm{INF}(m-1)$. □

2.4. $\frac{1}{q}\frac{p}{q}$-representation of r-adic numbers

Within this subsection letters $r, r_1, r_2, \dots$ stand for prime numbers and p is a general integer greater than one.

Definition 2.18. A left-infinite word $\cdots a_{-\ell_0+1}a_{-\ell_0}, \ell_0 \in \mathbb{N}$, over $\mathcal{A}_p$ is a $\frac{1}{q}\frac{p}{q}$*-representation* of $x \in \mathbb{Q}_r$ if $a_{-\ell_0} > 0$ or $\ell_0 = 0$ and

$$x = \sum_{k=-\ell_0}^{\infty} \frac{a_k}{q}\left(\frac{p}{q}\right)^k$$

with respect to $|\ |_r$.

So far, we have been concerned with $\frac{1}{q}\frac{p}{q}$-representation of rational numbers in $\mathbb{Q}_r$. We have shown that there exists at least one $\frac{1}{q}\frac{p}{q}$-representation for all rational numbers, namely the $\frac{1}{q}\frac{p}{q}$-expansion obtained by the MD algorithm, provided that r is a prime factor of p. Is this representation the only one of this type? Does it exist even for non-rational r-adic numbers? Before answering these questions, let us consider again Lemma 1.1: the number α_n is not the only integer satisfying the inequality; it remains true even if α_n is replaced by $\alpha_n + \ell r^n$ for any $\ell \in \mathbb{Z}$. This trivial observation turns out to be the reason why there exist even uncountably many $\frac{1}{q}\frac{p}{q}$-representations if p is not a power of a single prime. However, there is some common property for all such representations.

Lemma 2.19. *Let r be a prime factor of p with multiplicity i and let x be in $\mathbb{Q}_r$. Given $\cdots a_1 a_0 a_{-1} \cdots a_{-\ell_0}$, a_i in $\mathcal{A}_p$, such that*

$$x = \sum_{k=-\ell_0}^{\infty} \frac{a_k}{q}\left(\frac{p}{q}\right)^k,$$

then, for all integers $n \geqslant -\ell_0$,

$$\left| x - \sum_{k=-\ell_0}^{n} \frac{a_k}{q}\left(\frac{p}{q}\right)^k \right|_r \leqslant r^{-(n+1)i}. \tag{10}$$

Proof. The proof is again a consequence of the fact that $|\;|_r$ is ultrametric. We have

$$\left| x - \sum_{k=-\ell_0}^{n} \frac{a_k}{q}\left(\frac{p}{q}\right)^k \right|_r = \left| \sum_{k=n+1}^{\infty} \frac{a_k}{q}\left(\frac{p}{q}\right)^k \right|_r \leqslant \max_{k=n+1,n+2,\dots} \left| \frac{a_k}{q}\left(\frac{p}{q}\right)^k \right|_r \leqslant r^{-(n+1)i}.$$

$\square$

Having this necessary condition, we can characterize all $\frac{1}{q}\frac{p}{q}$-representations of a given x; the assumption x belongs to $\mathbb{Z}_r$ means no loss of generality.

Theorem 2.20. *Let r be a prime factor of p with multiplicity i and let x be in $\mathbb{Z}_r$.*

(i) *If p is not a power of r, then there exist uncountably many $\frac{1}{q}\frac{p}{q}$-representations $\cdots a_2 a_1 a_0$ such that for all $n \in \mathbb{N}$:*

$$\left| x - \sum_{k=0}^{n} \frac{a_k}{q}\left(\frac{p}{q}\right)^k \right|_r \leqslant r^{-(n+1)i}. \tag{11}$$

Each of these words is determined by an infinite sequence $(m_j)_{j\geqslant 0}$, $m_j \in \{0, 1, \dots, \tilde{r} - 1\}$, where $p = r^i \tilde{r}$.

(ii) *If p is a power of r, there exists a unique $\frac{1}{q}\frac{p}{q}$-representation satisfying* (11).

Proof. If $|x|_r \leqslant 1$, then $|qx|_r \leqslant 1$ as well. By Lemma 1.1, we know that there exits a unique $u_0 \in \{0, 1, \dots, r^i - 1\}$ such that $|qx - u_0|_r \leqslant r^{-i}$. Since the r-adic absolute value is ultrametric, we have for all $m \in \mathbb{N}$

$$|qx - (u_0 + mr^i)|_r \leqslant \max\{|qx - u_0|_r, |mr^i)|_r\} \leqslant r^{-i}.$$

Put $a_0 = u_0 + m_0 r^i$ for some $m_0 \in \{0, 1, \dots, \tilde{r} - 1\}$, then

$$|qx - a_0|_r = |x - \frac{a_0}{q}| \leqslant r^{-i} \quad \text{with } a_0 \in \mathcal{A}_p.$$

The integers a_0 of this form are the only integers of $\mathcal{A}_p$ satisfying this inequality.

Now, since $|1/p|_r = r^i$, multiplying the inequality by $|1/p|_r$ yields

$$\left| \frac{x - \frac{a_0}{q}}{p} \right|_r \leqslant 1$$

and so, as above, we have a unique $u_1 \in \{0, 1, \dots, r^i - 1\}$, arbitrary $m_1 \in \{0, 1, \dots, \tilde{r} - 1\}$ and $a_1 = u_1 + m_1 r^i$ such that

$$\left| q^2 \frac{x - \frac{a_0}{q}}{p} - u_1 \right|_r \leqslant r^{-i}.$$

Multiplying by $|p/q^2|_r = r^{-i}$ yields

$$\left| x - \frac{a_0}{q} - \frac{a_1}{q}\frac{p}{q} \right|_r \leqslant r^{-2i}.$$

In this way, after n steps, we obtain

$$\left| x - \sum_{k=0}^{n} \frac{a_k}{q} \left(\frac{p}{q}\right)^k \right|_r \leqslant r^{-(n+1)i}.$$

□

Theorem 2.20 provides an answer to the question of uniqueness of a representation and also characterizes all representations of $x \in \mathbb{Q}_r$ which converge to x with respect to $|\ |_r$. We have seen that for a rational x, which is an element of $\mathbb{Q}_r$ for all r prime, the $\frac{1}{q}\frac{p}{q}$-expansion $\langle x \rangle_{\frac{1}{q}\frac{p}{q}}$ converges with respect to all absolute values $|\ |_r, r$ a prime factor of p. So, it seems reasonable to study $\frac{1}{q}\frac{p}{q}$-representations which represent a rational x in $\mathbb{Q}_r$ for all r from any nonempty subset of prime factors of p.

Definition 2.21. Let $p = r_1^{\ell_1} \cdots r_k^{\ell_k}$ be a prime factorization of p, r_j are prime numbers > 1 and $\ell_j > 0$. Let $\mathbf{y} = (y_1, \cdots, y_k) \in \{0, \ell_1\} \times \cdots \times \{0, \ell_k\} \setminus (0, 0, \ldots, 0)$. We denote $r^{\mathbf{y}} = r_1^{y_1} \cdots r_k^{y_k}$, $I(\mathbf{y}) = \{j \mid y_j = \ell_j\}$, and $\widetilde{r^{\mathbf{y}}}$ is defined by $p = r^{\mathbf{y}}\widetilde{r^{\mathbf{y}}}$.

Algorithm 2.22 (Generalized Modified Division (GMD) Algorithm)**.** *Let* $\mathbf{y}$ *as in Definition 2.21 be fixed but arbitrary for a given* p *and* $x = \frac{s}{t} \in \mathbb{Q}$ *such that* $t > 0$ *is co-prime to* r_j *for all* $j \in I(\mathbf{y})$*. Put* $s_0 = s, t_0 = t$*. Moreover let*

$$t_j = t_{j-1}\widetilde{r^{\mathbf{y}}} = t_0(\widetilde{r^{\mathbf{y}}})^j \quad \text{and} \quad q\frac{s_j}{t_j} = \frac{s'_{j+1}}{t_j} r^{\mathbf{y}} + u_j \quad \text{with } u_j \in \{0, 1, \ldots, r^{\mathbf{y}} - 1\}.$$

Choose $m_j \in \{0, 1, \ldots, \widetilde{r^{\mathbf{y}}} - 1\}$ *at random and put* $a_j = u_j + m_j r^{\mathbf{y}}$ *and* $s_{j+1} = s'_{j+1} - m_j t_j$*. Return* $\cdots a_2 a_1 a_0$.

Denote the set of all possible outputs $\cdots a_2 a_1 a_0$ *by GMD*(x)*.*

We now prove that the GMD algorithm returns all $\frac{1}{q}\frac{p}{q}$-representations of x in $\mathbb{Q}_{r_j}, j \in I(\mathbf{y})$.

Lemma 2.23. *Given* p *and* $\mathbf{y}$*. Let* $x = \frac{s}{t} \in \mathbb{Q}$ *such that* $t > 0$ *is co-prime to* r_j *for all* $j \in I(\mathbf{y})$*. There exist exactly* $\widetilde{r^{\mathbf{y}}}$ *numbers* $a \in \mathcal{A}_p$ *satisfying* $|x - a|_{r_j} \leqslant r^{-\ell_j}$ *for all* $j \in I(\mathbf{y})$*.*

Proof. The existence of $\widetilde{r^{\mathbf{y}}}$ such numbers follows from the construction of the GMD algorithm: $c_m = u_0 + mr^{\mathbf{y}}, m = 0, 1, \ldots, \widetilde{r^{\mathbf{y}}} - 1$ satisfy the inequality, $0 \leqslant u_0 \leqslant r^{\mathbf{y}} - 1$ is the letter constructed in the first step of the GMD algorithm. If there is another number in $\mathcal{A}_p$ different from all c_m and satisfying the inequality, there must exist $\widetilde{r^{\mathbf{y}}}$ such digits of the form of $d_m = d_0 + mr^{\mathbf{y}}, m = 0, 1, \ldots, \widetilde{r^{\mathbf{y}}} - 1$ with $0 \leqslant d_0 \leqslant r^{\mathbf{y}} - 1$.

As we know, for all $j \in I(\mathbf{y})$, there exists a unique $0 \leqslant b \leqslant r^{\ell_j} - 1$ such that $|x - b|_{r_j} \leqslant r^{-\ell_j}$ and, furthermore, all other numbers for which this inequality is true are of the form of $b + nr^{\ell_j}, n \in \mathbb{Z}$. Thus, both c_0 and d_0 are of this form and so $c_0 - d_0$ is a multiple of r^{ℓ_j} for all $j \in I(\mathbf{y})$. Since r_j are distinct primes, $c_0 - d_0$ must be a multiple of $r^{\mathbf{y}}$ and hence $c_0 = d_0$. □

Theorem 2.24. *Let* $\mathbf{y}$ *as in Definition 2.21 be fixed but arbitrary for a given* p *and* $x = \frac{s}{t} \in \mathbb{Q}$ *such that* $t > 0$ *is co-prime to* r_j *for all* $j \in I(\mathbf{y})$*. Further, let* $\mathbf{a} = \cdots a_2 a_1 a_0$ *be an infinite word over* $\mathcal{A}_p$*.*

Then

$$x = \sum_{k=0}^{\infty} \frac{a_k}{q} \left(\frac{p}{q}\right)^k$$

with respect to $|\ |_{r_j}$ *for all* $j \in I(\mathbf{y})$ *if, and only if,* $\mathbf{a} \in GMD(x)$.

Proof. First, let us suppose $\mathbf{a} \in \mathrm{GMD}(x)$. We have

$$x = \frac{s_0}{t_0} = \frac{s_1}{t_0}\frac{r^{\mathbf{y}}}{t_0} + \frac{a_0}{q} = \frac{s_1}{t_1}\frac{p}{q} + \frac{a_0}{q} = \frac{p}{q}\left(\frac{s_2}{t_2}\frac{p}{q} + \frac{a_1}{q}\right) + \frac{a_0}{q} =$$
$$= \left(\frac{p}{q}\right)^2\left(\frac{s_3}{t_3}\frac{p}{q} + \frac{a_2}{q}\right) + \frac{p}{q}\frac{a_1}{q} + \frac{a_0}{q} = \cdots = \frac{s_{n+1}}{t_{n+1}}\left(\frac{p}{q}\right)^{n+1} + \sum_{k=0}^{n}\frac{a_k}{q}\left(\frac{p}{q}\right)^k.$$

Hence, for all $n \in \mathbb{N}$ and for all $j \in I(\mathbf{y})$

$$\left|x - \sum_{k=0}^{n}\frac{a_k}{q}\right|_{r_j} = \left|\frac{s_{n+1}}{t_{n+1}}\left(\frac{p}{q}\right)^{n+1}\right|_{r_j} = \left|\frac{s_{n+1}}{t_0(\widetilde{r^{\mathbf{y}}})^{n+1}}\right|_{r_j}\left|\left(\frac{p}{q}\right)^{n+1}\right|_{r_j}.$$

Since $t_0(\widetilde{r^{\mathbf{y}}})^{n+1}$ is co-prime to r_j and $|s_{n+1}|_{r_j} \leqslant 1$, $r_j^{-(n+1)\ell_j}$ is an upper bound and the sum converges to x.

Assume that

$$x = \sum_{k=0}^{\infty}\frac{a_k}{q}\left(\frac{p}{q}\right)^k$$

with respect to $|\ |_{r_j}$ for all $j \in I(\mathbf{y})$. Then $|qx - a_0|_{r_j} \leqslant {r_j}^{-\ell_j}, \quad j \in I(\mathbf{y})$. The previous lemma says that there are just $\widetilde{r^{\mathbf{y}}}$ possible values of a_0, and so they must coincide with the $\widetilde{r^{\mathbf{y}}}$ values of the first digit (possibly) obtained in the first step of the GMD algorithm.

Since again

$$\left|\frac{q^2}{p}\left(x - \frac{a_0}{q}\right) - a_1\right|_{r_j} \leqslant {r_j}^{-\ell_j}, \quad j \in I(\mathbf{y}),$$

we can use the same argument for a_1 and continue in the same manner for $a_2, a_3, \ldots$. □

Obviously, if we take $\mathbf{y} = (\ell_1, \ell_2, \ldots, \ell_k)$, then the GMD and MD algorithms coincide and the returned word is unique and equal to $\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$.

Example 2.25. Let $p = 30$, $q = 11$, and $\mathbf{y} = (1, 1, 0)$. Here are two examples of representations of the number 1:

$$\begin{array}{lcl} \cdots 27\ 24\ 24\ 29\ 26\ 29\ 27\ 25\ 25\ 24\ 28\ 24\ 28\ 27\ 29 & \in & \mathrm{GMD}(1), \\ \cdots 20\ 22\ 21\ 22\ 22\ 22\ 19\ 18\ 18\ 19\ 23\ 18\ 22\ 22\ 23 & \in & \mathrm{GMD}(1), \end{array}$$

and of the number 0:

$$\begin{array}{lcl} \cdots\ 7\ \ 2\ \ 3\ \ 7\ \ 4\ \ 7\ \ 8\ \ 7\ \ 7\ \ 5\ \ 5\ \ 6\ \ 6\ \ 5\ \ 6 & \in & \mathrm{GMD}(0), \\ \cdots 15\ 12\ 11\ 15\ 11\ 13\ 15\ \ 9\ \ 9\ \ 9\ 14\ 12\ 12\ 10\ 12 & \in & \mathrm{GMD}(0). \end{array}$$

2.5. Periodicity

It turns out that the $\frac{1}{q}\frac{p}{q}$-expansion $\langle x\rangle_{\frac{1}{q}\frac{p}{q}}$ of a rational x plays an important role between all representations from $\mathrm{GMD}(x)$ not only because it is the only one which converges in all $\mathbb{Q}_r, r$ a prime factor of p. It is also the only one which is eventually periodic.

Theorem 2.26. *Let* $x \in \mathbb{Q}_{r_i}, r_i > 1$ *a prime factor of* $p = r_1^{\ell_1} \cdots r_k^{\ell_k}$. *Then the* $\frac{1}{q}\frac{p}{q}$*-representation* $\mathbf{a}$ *of* x *is eventually periodic if, and only if,* $x \in \mathbb{Q}$ *and* $\mathbf{a} = \langle x\rangle_{\frac{1}{q}\frac{p}{q}}$.

Proof. The right to left implication is proved in Lemma 2.3.

Let us assume, w.l.o.g, that $x \in \mathbb{Z}_r$ and that $\mathbf{a} = \cdots a_2 a_1 a_0 \in \mathrm{GMD}(x)$ for some $\mathbf{y}$ (see Definition 2.21) is eventually periodic, say, $\mathbf{a} = {}^{\omega}wv, w \in \mathcal{A}_p^+, v \in \mathcal{A}_p^*$, $|v| = h_0$, and $|w| = h$. The simple fact

$$\sum_{j=0}^{\infty} \left(\frac{p}{q}\right)^j = \frac{1}{1 - \frac{p}{q}}$$

immediately implies that x must be rational.

Let $(s_j)_{j \geqslant 1}$ and $(t_j)_{j \geqslant 1}$ be the sequences constructed within the run of the GMD algorithm. Then we must have for all $n \geqslant h_0$ and $j \in \mathbb{N}$

$$\frac{s_n}{t_n} = \frac{s_{n+jh}}{t_{n+jh}},$$

since $t_j = t_0 \widetilde{r^{\mathbf{y}}}^j$, we get $s_{n+jh} = s_n (\widetilde{r^{\mathbf{y}}})^{jh}$ which implies $|s_{n+jh}|_{r_i} \leqslant r_i^{-jh}$ for all $i \in \{1, 2, \ldots, k\} \setminus I(\mathbf{y})$. Define for all $n \geqslant h_0$ and for all these i a nonnegative integer $m_0 < h$ by $n - h_0 \equiv m_0 \pmod h$, then

$$\left| x - \sum_{j=0}^{n-1} \frac{a_j}{q} \left(\frac{p}{q}\right)^j \right|_{r_i} = \left| \frac{s_n}{t_n} \left(\frac{p}{q}\right)^n \right|_{r_i} = \left| \frac{s_{h_0+m_0} (\widetilde{r^{\mathbf{y}}})^{n-h_0-m_0}}{t_0 (\widetilde{r^{\mathbf{y}}})^n} \left(\frac{p}{q}\right)^n \right|_{r_i} = \left| \frac{s_{h_0+m_0}}{t_0} \right|_{r_i} r_i^{-(n-h_0-m_0)}.$$

Since the integer m_0 is bounded by h for arbitrary n, the sum converges to x with respect to all absolute values $|\ |_{r_i}, i \in \{1, 2, \ldots, k\}$. But there is only one such $\frac{1}{q}\frac{p}{q}$-representation: namely, $\mathbf{a}$ must be equal to $\langle x \rangle_{\frac{1}{q}\frac{p}{q}}$. □

3. Converters between rational base number systems

3.1. $\frac{p}{q}$-representations

Let us now consider the system corresponding to the first series from (4). In the same way as for the AFS system, we define the $\frac{p}{q}$-*expansion* of x in $\mathbb{Q}$ and $\frac{p}{q}$-*representation* of x in $\mathbb{Q}_r$. As promised in the introduction, we show that there exists a simple converter between them. A *transducer* is an automaton where edges are labelled by couples of words. It is finite if the set of states and the set of edges are finite. It is said to be *sequential* if the projection on the first component is a deterministic automaton. It is *letter-to-letter* if edges are labelled by couples of letters. For more definitions and results on transducers the reader is referred to [9] for instance.

Theorem 3.1. *There exists a finite letter-to-letter right*[2] *sequential transducer $\mathcal{C}$ converting the $\frac{p}{q}$-representation of any $x \in \mathbb{Z}_r, r$ prime factor of p, to its $\frac{1}{q}\frac{p}{q}$-representation; the inverse of $\mathcal{C}$ is also a finite letter-to-letter right sequential transducer.*

Proof. Let $\mathcal{C} = (Q \subset \mathbb{N}, \mathcal{A}_p \times \mathcal{A}_p, E, \{0\}, \omega)$ be the right letter-to-letter sequential transducer whose set of edges E is defined by $s \xrightarrow{a|b} s' \quad \Leftrightarrow \quad q(a+s) = ps' + b$ with $a, b \in \mathcal{A}_p$.

Clearly, if the input is $\cdots a_1 a_0$ such that $x = \sum_{i=0}^{\infty} a_i \left(\frac{p}{q}\right)^i$ in $\mathbb{Q}_r$, r a prime factor of p, then there is in $\mathcal{C}$ a path $0 \xrightarrow{a_0|b_0} s_1 \xrightarrow{a_1|b_1} s_2 \cdots$ such that, for each $k \geqslant 0$,

$$\sum_{i=0}^{k} a_i \left(\frac{p}{q}\right)^i = \sum_{i=0}^{k} \frac{b_i}{q} \left(\frac{p}{q}\right)^i + s_{k+1} \left(\frac{p}{q}\right)^{k+1}.$$

[2]Words are processed from right to left.

The states s_i are nonnegative integers, and so

$$\sum_{i=0}^{\infty} a_i \left(\frac{p}{q}\right)^i = \sum_{i=0}^{\infty} \frac{b_i}{q} \left(\frac{p}{q}\right)^i \quad \text{in } \mathbb{Q}_r.$$

Let us show that $\mathcal{C}$ is finite. From a state s, it is possible to reach the state $s+k, k \geqslant 1$, if there exist a and b in $\mathcal{A}_p$ such that $q(s+a) = p(s+k)+b$, that is, if $s = \frac{qa-kp-b}{p-q}$. Since $a \leqslant p-1, b \geqslant 0$, the largest accessible state is

$$s_{\max} \leqslant \max_{k\geqslant 1} \left\{ \frac{q(p-1)-kp}{p-q} + k \right\} \leqslant \max_{k\geqslant 1} \left\{ \frac{q(p-1)}{p-q} - k \underbrace{\left(\frac{p}{p-q} - 1\right)}_{>0} \right\} = \frac{qp}{p-q}$$

and hence the transducer $\mathcal{C}$ is finite.

Now, suppose that $s \xrightarrow{a|b} s'$ and $s \xrightarrow{a'|b} s''$, $a \neq a'$. Then $q(a+s)$ and $q(a'+s)$ are congruent (mod p), hence a and a' are congruent (mod p), which is impossible. Thus, the transducer $\tilde{\mathcal{C}}$, where the edges $\tilde{E}$ are defined by

$$s \xrightarrow[\tilde{\mathcal{C}}]{a|b} s' \quad \Leftrightarrow \quad s \xrightarrow[\mathcal{C}]{b|a} s',$$

is also right sequential, and $\tilde{\mathcal{C}}$ realizes the conversion from $\frac{1}{q}\frac{p}{q}$-representations to $\frac{p}{q}$-representations. $\square$

This result says that there is a one-to-one mapping between the sets of all $\frac{p}{q}$- and $\frac{1}{q}\frac{p}{q}$-representations of a given $x \in \mathbb{Z}_r$. This mapping, moreover, preserves eventual periodicity, meaning that the eventually periodic infinite words are mapped to $\frac{1}{q}\frac{p}{q}$-expansions. This is not a surprising result as it is still true that only rational numbers can have an eventually periodic $\frac{p}{q}$-representation.

Regarding the finiteness of the expansions, there is a difference. But finding those rationals with finite $\frac{p}{q}$-representations can be done in the perfectly analogous way we used for the $\frac{1}{q}\frac{p}{q}$ case in Proposition 2.17.

Theorem 3.1 can be easily modified also for the two negative base systems from (4). Since the composition of two finite sequential transducers is again a finite sequential transducer, the theorem is valid for any pair of number systems form (4). This conversion still preserves eventual periodicity. The question on finiteness for the negative base cases is a bit more complex. The two systems with negative rational base are *canonical number systems* (see [2] for more), *i.e.*, each x in $\mathbb{Z}$ has a unique finite representation, but there are also rational numbers with finite representations.

3.2. Conversion from the integer base system

Another natural question is whether there exists a converter of representations in integer base p to $\frac{1}{q}\frac{p}{q}$-representations. The answer is positive, but the converter is not finite. This is an expected result since if there existed such a finite converter, there would be a finite converter from the language $\mathcal{A}_p^*$ of standard positive integer representations to the non-context-free language $L_{\frac{1}{q}\frac{p}{q}}$, which is not possible.

Algorithm 3.2. *Denote by* $\cdots a_1 a_0 \in {}^{\mathbb{N}}\!\mathcal{A}_p$ *the input and by* $\cdots b_1 b_0 \in {}^{\mathbb{N}}\!\mathcal{A}_p$ *the output. The rewriting rule is defined by:* $z_0 = 0, i = 0$ *and* $(z_i, i) \xrightarrow{a_i|b_i} (z_{i+1}, i+1)$, *with* $a_i, b_i \in \mathcal{A}_p$ *such that*

$$a_i q^i + z_i = \frac{b_i}{q} + \frac{p}{q} z_{i+1}.$$

Clearly, z_i is always nonnegative and uniquely given.

Proposition 3.3. *Let r be a prime factor of p and let $\cdots a_1 a_0 \in {}^{\mathbb{N}}\mathcal{A}_p$ such that $x = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_r$. Then for the output $\cdots b_1 b_0 \in {}^{\mathbb{N}}\mathcal{A}_p$ of Algorithm 3.2 we have $x = \sum_{i=0}^{\infty} \frac{b_i}{q}\left(\frac{p}{q}\right)^i \in \mathbb{Z}_r$.*

Proof. The proof is simple, it follows from the fact that

$$z_0 = 0 = \frac{b_0}{q} + \frac{p}{q} z_1 - a_0 = \frac{b_0}{q} + \frac{b_1}{q}\frac{p}{q} + \left(\frac{p}{q}\right)^2 z_2 - pa_1 - a_0 = \cdots$$
$$= \sum_{i=0}^{k} \frac{b_i}{q}\left(\frac{p}{q}\right)^i + \left(\frac{p}{q}\right)^{k+1} z_{k+1} - \sum_{i=0}^{k} a_i p^i$$

for all $k \in \mathbb{N}$ and from that
$$\left|\left(\frac{p}{q}\right)^{k+1} z_{k+1}\right|_r \longrightarrow 0 \quad \text{as } i \to \infty.$$
□

This conversion preserves finiteness. Note that Algorithm 3.2 allows to define a letter-to-letter right sequential transducer with a denumerable set of states realizing the conversion.

Lemma 3.4. *If the input of Algorithm 3.2 is finite (i.e., eventually zero), then the output is finite as well.*

Proof. Let the input be equal to $\cdots a_1 a_0 \in {}^{\mathbb{N}}\mathcal{A}_p$ where $a_i = 0$ for all $i > k \in \mathbb{N}$. Then for all $j \in \mathbb{N}$

$$z_{k+j+1} = \frac{q}{p}\left(z_{k+j} - \frac{b_{k+j}}{q}\right) < z_{k+j} \quad \text{if } z_{k+j} \neq 0 \text{ or } b_{k+j} \neq 0.$$

Thus, the sequences $(z_i)_{i \geqslant 0}$ and $(b_i)_{i \geqslant 0}$ must be eventually zero. □

The input is finite if it is a representation of a nonnegative integer in base p. This implies that the lemma cannot be reversed since, as we know, there are finite outputs obtained for infinite inputs (a trivial example is the representations of $\frac{p}{q}$).

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