Equational presentations of tree-automatic structures

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Abstract

We investigate in this abstract various possibilities for representing tree-automatic structures. In particular we show that the classical presentation of prefix-recognizable structures as least solutions of equational systems admit a natural extension for tree-automatic structures. We also show that the first-order logic extended with counting quantifiers remains decidable for the solutions of some infinite equational systems. This extends the decidability results known for tree-automatic structures.

1 Introduction

Tree automatic structures were introduced in [2] though the underlying idea can be traced back to the work of Dauchet and Tison [8]. Tree automatic structures are a natural extension of automatic structures [9] over an universe made of trees. More precisely, in a tree-automatic structure, the universe is a rational set of finite terms, and the interpretation of each relational symbol is automatic in the meaning that it is described by a finite state automaton. By extension, we say (in this abstract) that a structure is tree-automatic whenever it is isomorphic to a tree-automatic structure. The definition of automaticity of a relation is such that automatic relations are closed by the boolean connectives as well as projection and cylindrification. This closure properties result in a decidable first-order theory for tree-automatic structures. In fact this result can be strenghtens to capture first-order logic extended with counting quantifiers[1].

Automatic structures belong to a more general topic, the study of the classes of structures admitting a finite presentation, i.e. that each (possibly infinite) structure can be described by a finite object. A widely studied class of this kind is the one of prefix-recognizable structures [4]. It happens that each structure in this class has different presentations. Let us cite three such presentations. The first one is the *internal presentation*, in which an exact description of the universe and of the relation is given: for prefix-recognizable structures, the universe is a rational set of words and each relation is described by a regular set of prefix rewriting rules[4]. The second way is *transformational* as each structure is described by a transformation applied to a given known structure; in this sens, the prefix-recognizable structures are the monadic (second-order) interpretations of the infinite complete binary tree[1, 3]. The third presentation is *equational*; in this sens, the prefix-recognizable structures are the least solutions of finite equational systems over a given set of fixed operators called VR (standing for vertex replacement) [1].

In this abstract, we present similar results concerning the class of tree-automatic structures. Those structures classically admit an internal presentation. We mainly contribute here with an equational presentation: we provide a set of operators such that a structure is tree-automatic iff it is isomorphic to the solution of an finite equational system using those operators (Theorem 1). This approach unifies tree-automatic structures with prefix-recognizable ones since the operators we use are the ones defining the prefix recognizable structures, but enhanced with a product operator. It also allows us to use tools specially designed for the treatment of equational systems, such as tree transducers with lookahead [6, 5] for the study of tree-automatic structures. Finally, it is natural in this context to step outside the case of finite equational systems and consider infinite such systems. Theorem 2 shows that in this case, some representation equivalences remain true. Corollary 3 then provides decidability results for those "extended tree-automatic structures".

Detailed proofs can be found in [5].

[1]The counting quantifiers are $\exists^{\omega} x$ and $\exists^{m[n]} x$ meaning respectively "there exists infinitely many x's such that ..." and "there exists m-many x's modulo n such that ...".

2 Equational presentation: the VRC operators

The core of the VRC operators that we are about to introduce is the positive quantifier free definable (pqfd) interpretation. Formally a pqfd interpretation $\mathcal{I}$ is an operation which transforms relational structures into relational structures and is described by a tuple of formulas $(\delta, \phi_{R_1}, \ldots, \phi_{R_n})$ (where $R_1, \ldots, R_n$ are the relational symbols of the resulting structure). In this case, each formula is made only of predicates applied to first-order variables, conjonctions, disjonctions and the constants true and false (i.e. first-order without quantification and without negation). As usually, the formula δ has a single free variable and is used to define the universe of the resulting structure, and each formula ϕ_R has $\mathbf{arity}(R)$ free variables and defines the new interpretation of the relational symbol R.

There is four kind of *VRC operators*: the constant structure with a singleton universe and no relation; a binary operator performing the *disjoint union* of structures; a binary operator performing the *cartesian product* of structures; and finally, a unary operator corresponding to each *pqfd interpretation*.

All these operators have the property of being continuous (in the meaning e.g. of ω-complete partial orders), and thus by the theorem of Knaster-Tarski, each equational system using them admits a unique least solution. In particular, it is known from [1] that the solution of finite equational systems using all VRC operators but the product, are isomorphic to prefix-recognizable structures. The following theorem establishes a similar result for tree-automatic structures.

Theorem 1 *A structure is tree-automatic iff it is isomorphic to the least solution of a finite equational system over the VRC operators.*

3 Powerset monadic-interpretation

We provide here an transformational presentation suitable for representing the solutions of the (possibly infinite) equational systems over the VRC operators.

A *powerset monadic-interpretation* $\mathcal{I}^{\mathcal{P}}$ is described by a tuple of formulas $(\delta, \phi_{R_1}, \ldots, \phi_{R_n})$, where this time the formulas are monadic and the free variables are *monadic*. Applied to a structure, this interpretation produces a new structure where the elements of the universe are the subsets of the first universe satisfying δ. Similarly, the interpretation of the relations is defined via the ϕ formulas. It is easy to see that if the original structure has a deciable monadic theory, the one resulting of a powerset interpretation has a decidable first-order theory. In fact, one has the following refinement: when the original structure is a deterministic tree, then the resulting structure has a decidable first-order theory with counting quantifiers.

The following theorem shows that solving an equational system over the VRC operators is equivalent to applying a powerset monadic-interpretation to an infinite determinisic tree. In this theorem, we see equational systems as deterministic graphs containing, correctly encoded, all the relevant information.

Theorem 2 *The following class of structures are equivalent,*

- *the least solutions of equational systems using a finite subset of the VRC operators,*
- *the countable powerset monadic-interpretations of the unravelling of a deterministic graph.*

And this equivalence is effective in the meaning that the graphs and the equational systems are linked by parameterless MSO-definable transductions (see [7]).

Combined with the decidability remarks mentionned above, we obtain the following corollary.

Corollary 3 *If the monadic theory of an equational system using a finite subset of the VRC operators is decidable then the first-order theory with counting quantifiers of its least solution is decidable.*

Let us notice that in this corollary, the classical decidability result for tree-automatic structures simply corresponds to the particular case of finite equational systems.

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