Factorisation forests for infinite words

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Cnrs, Irisa
Fct 2007

OVERVIEW

- Factorisation forest theorems
  - Original presentation, by trees
  - By Ramseyan regular expressions
  - By splits
- An elementary application
- Ideas about the proof
- Extension to infinite words
- Application to automata over scattered linear orderings
- Related work

Semigroups

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E.g., $(A^*, .)$ is a semigroup.
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E.g., Words over $\{a, b\}$ containing an even number of a's is $f^{-1}(0)$ with
Semigroup: $S = \{0, 1\}$, $00 = 11 = 0$, $10 = 01 = 1$
Morphism: $f(a) = 1$, $f(b) = 0$

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Below we identify S with the alphabet (f is the identity).
I.e., we work with words in S^*.
Call $f(u)$ the value of u.

Factorisation theorem by an example

Set $S = (\mathbb{Z}/5\mathbb{Z}, +)$ and the word $u =$210232300322002.

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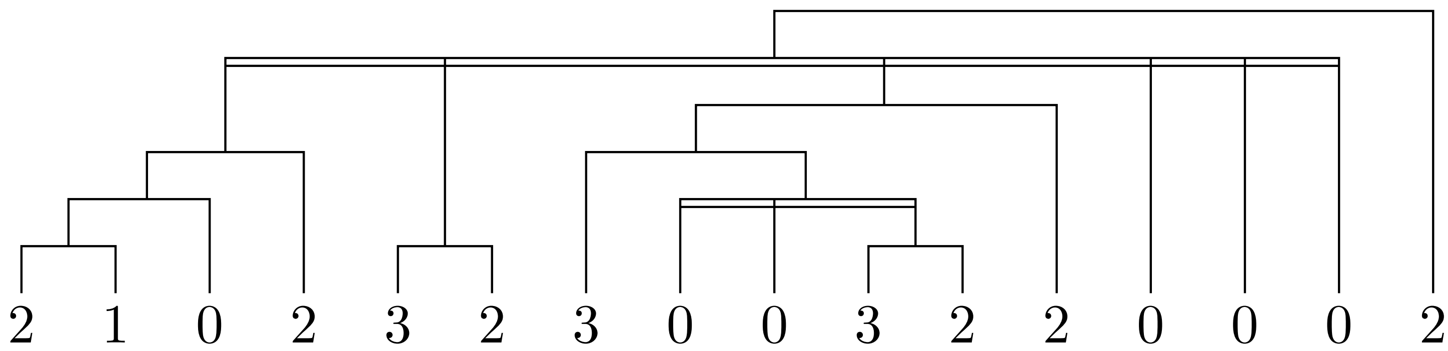

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- leaves are labeled by letters,
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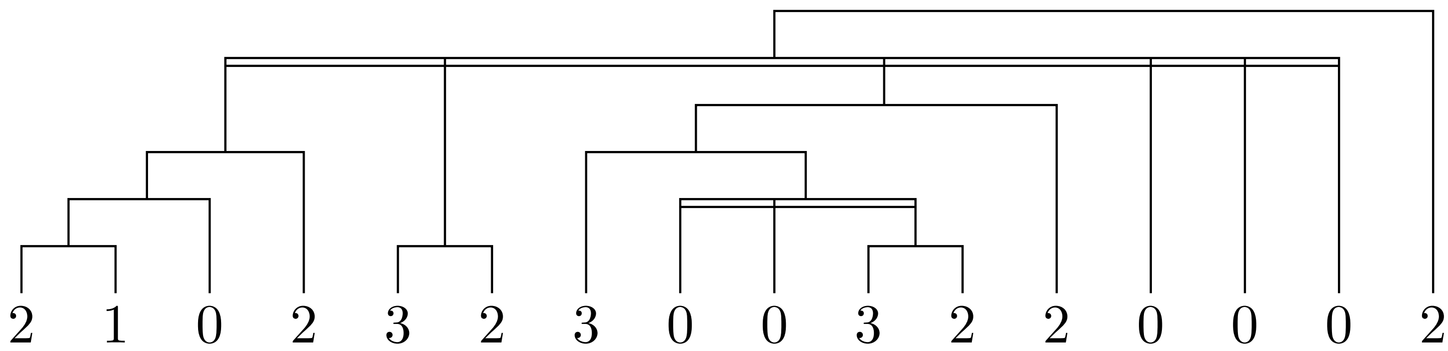

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A factorising tree is Ramseyan if every node:

- is a leaf, or;
- has two children, or;
- all its children have as value the same idempotent of S.

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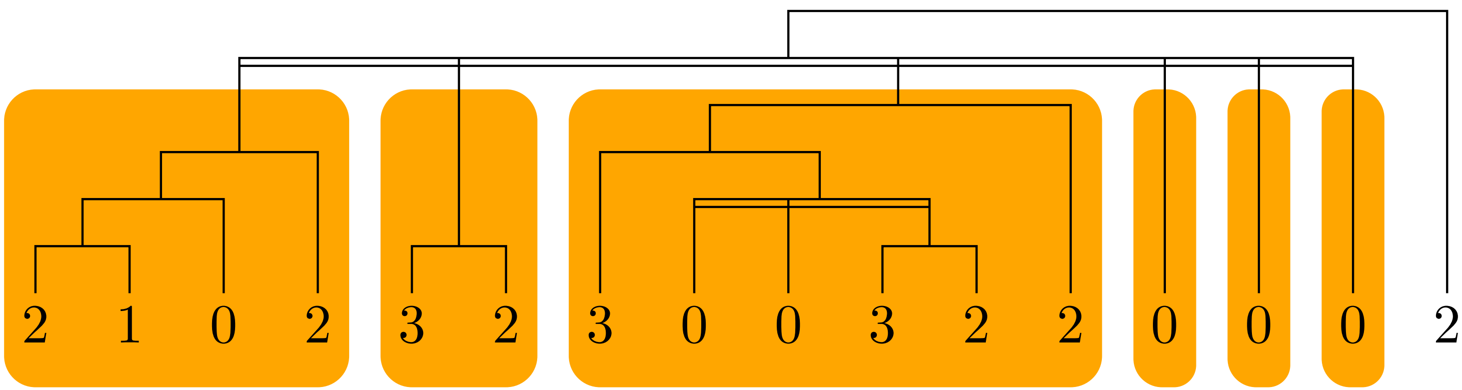

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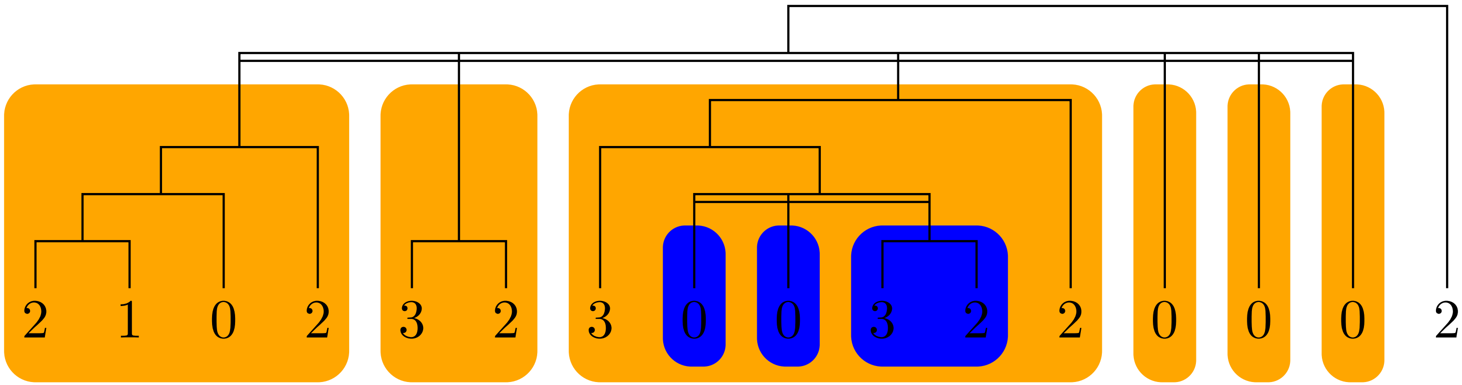

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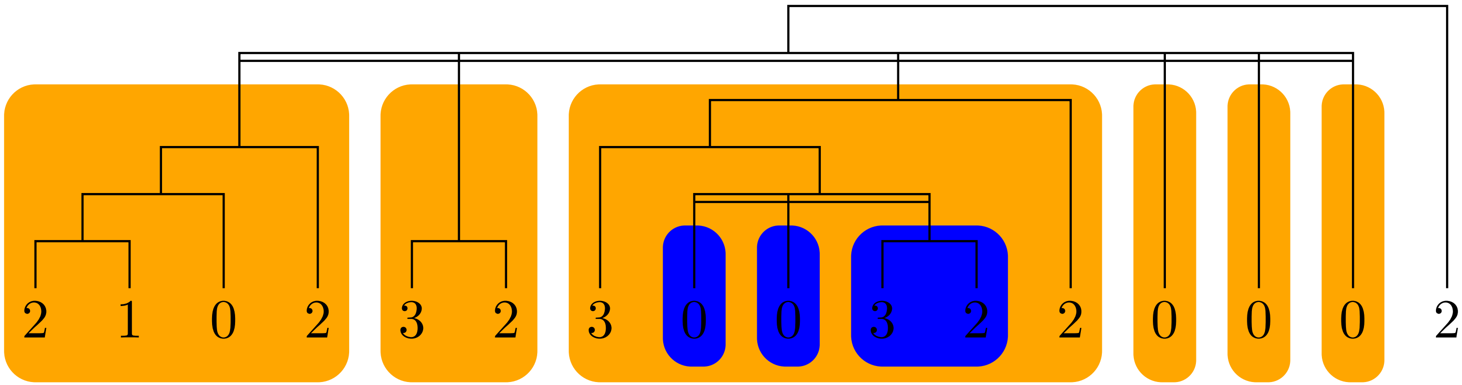

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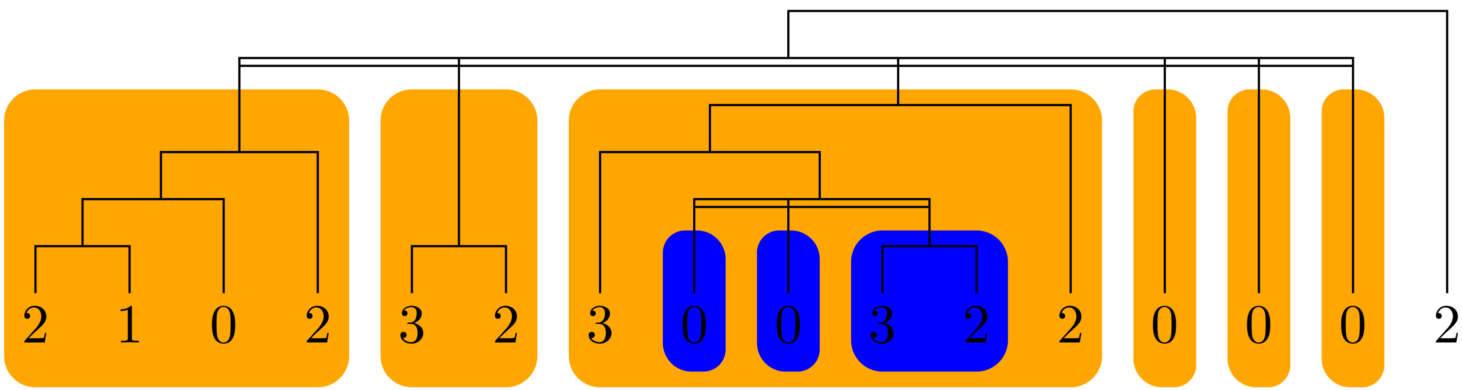

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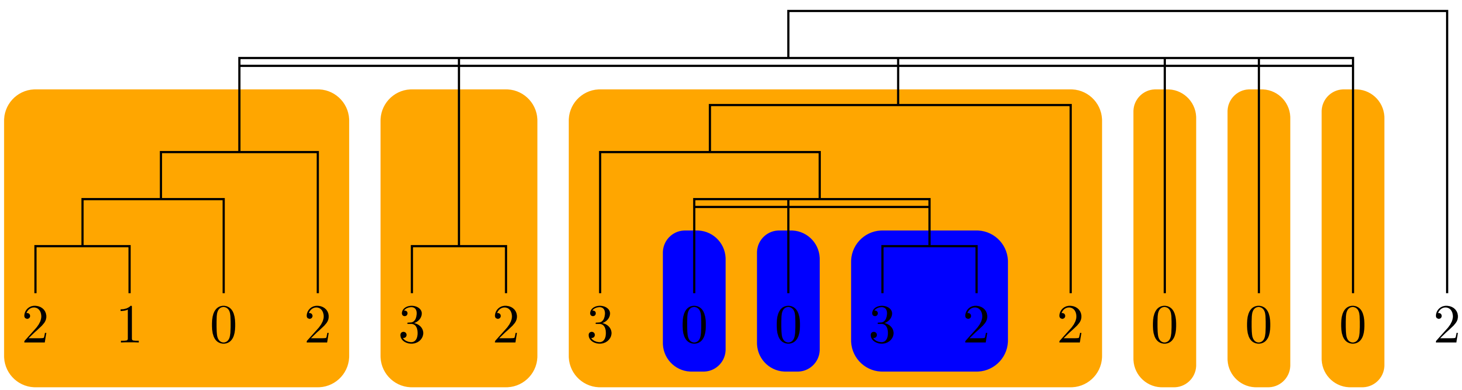

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STATEMENT BY REGULAR EXPRESSION

Th: For every finite semigroup S, there exists a regular expression such that:

- it evaluates to S^*
- Kleene star L^* is allowed only if all the values of words in L are equal to the same idempotent

E.g., $S = (\mathbb{Z}/2\mathbb{Z}, +)$

$$0^*(10^*10^*)^*.(\varepsilon + 10^*)$$

Statement by splits

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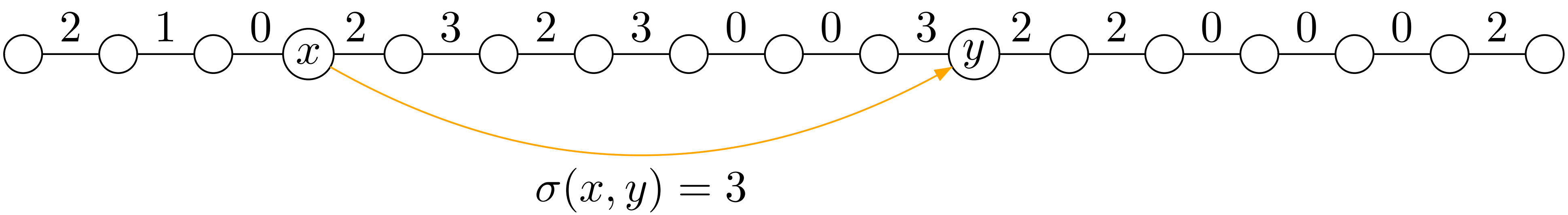

Def: Let α be a linear ordering, an additive labeling $\sigma : \alpha^2 \to S$ is such that:

- $\sigma(x, y)$ is defined iff $x < y$, and,
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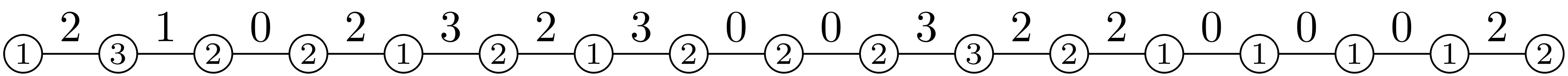

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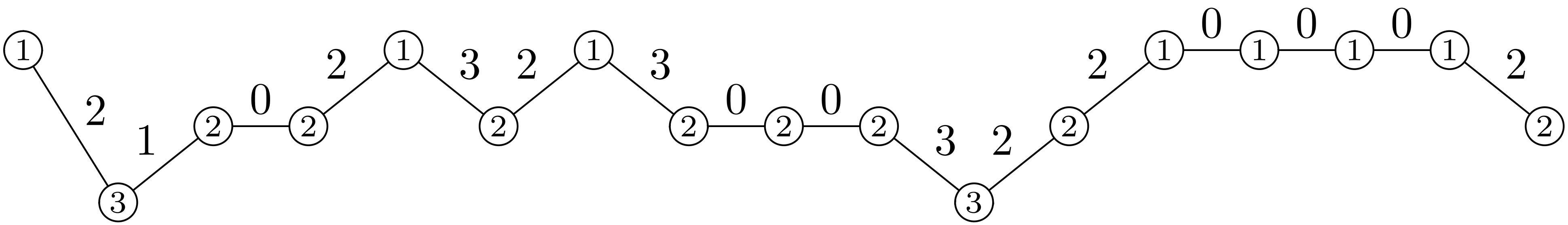

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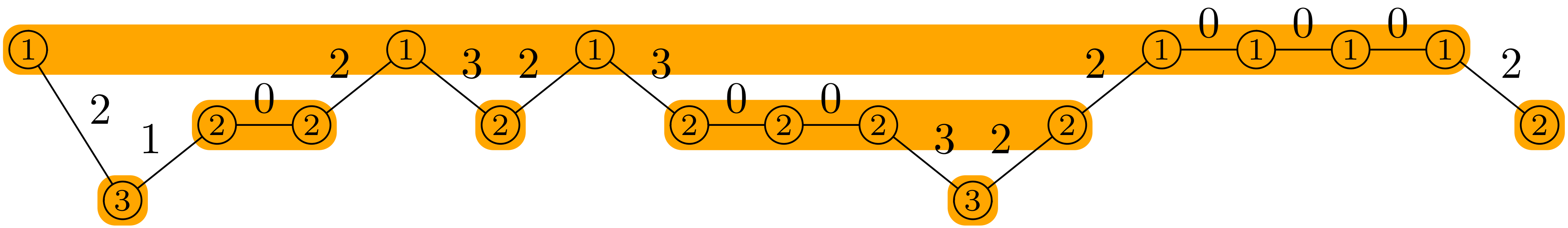

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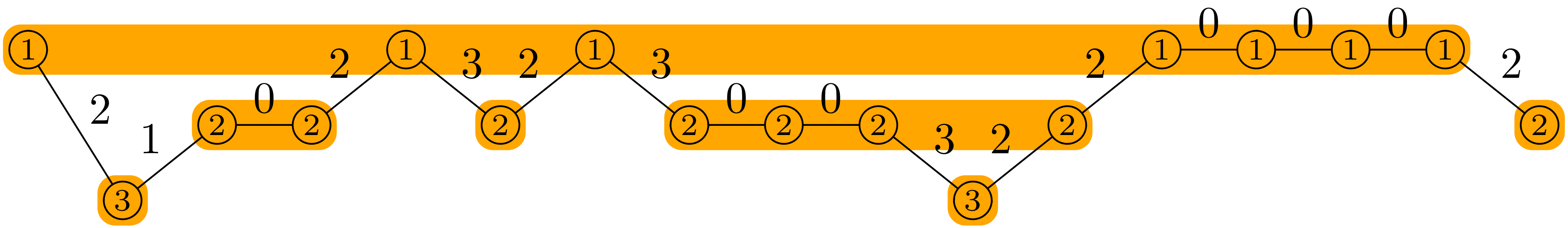

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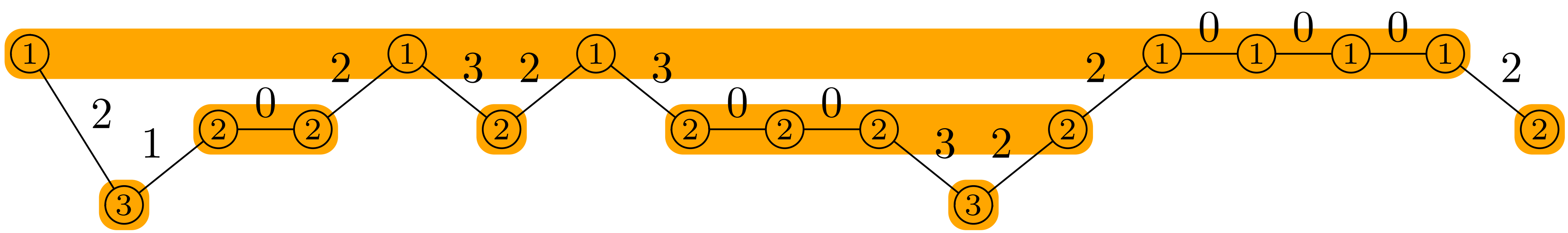

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Th: Every additive labeling of a finite linear ordering by a finite semigroup S admits a Ramseyan split of height at most $|S|$.

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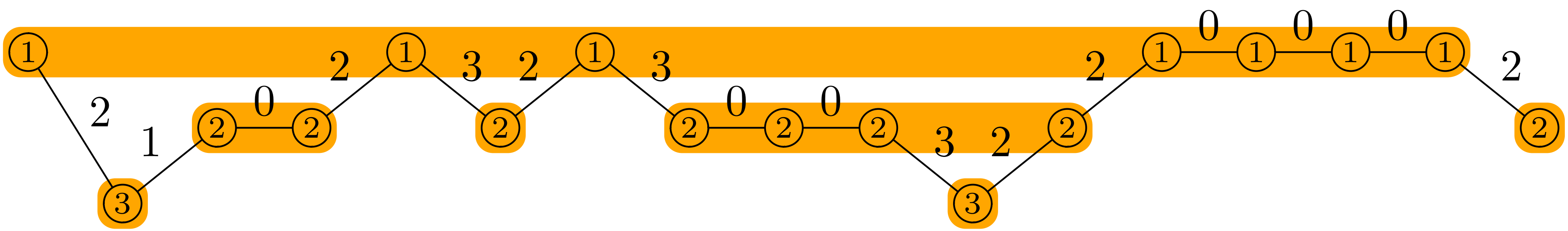

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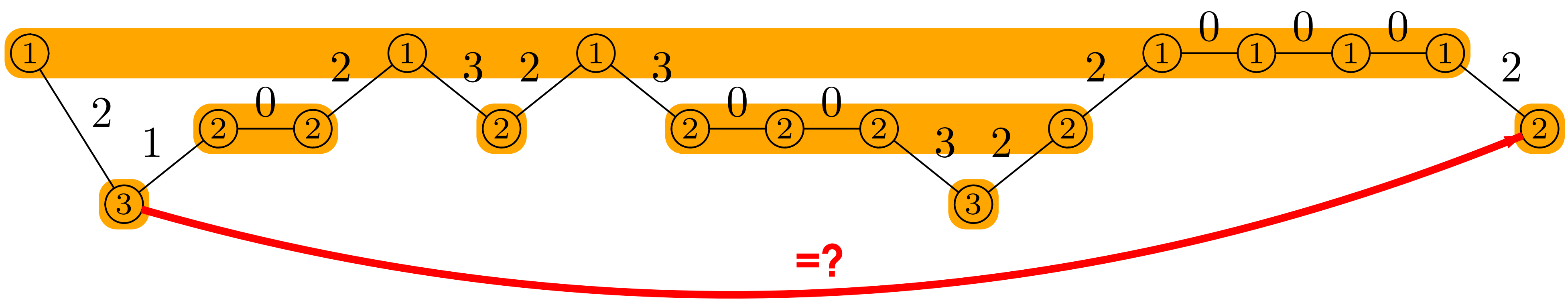

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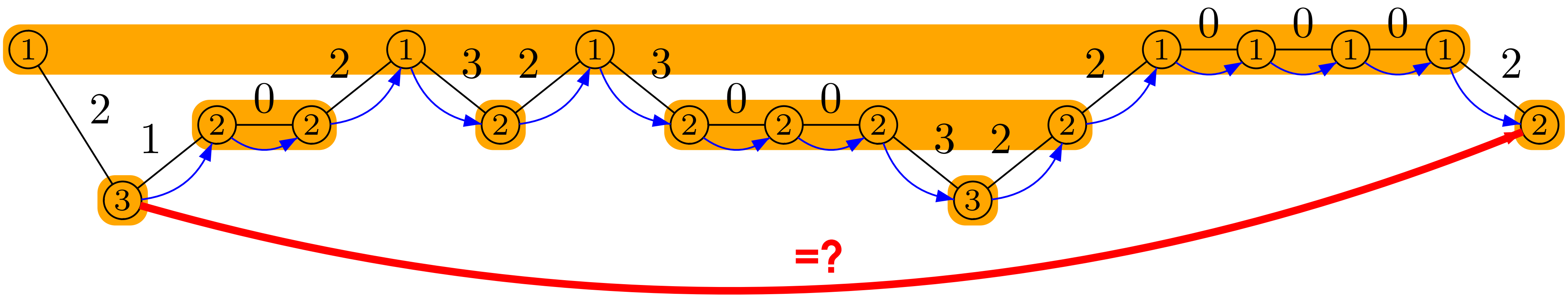

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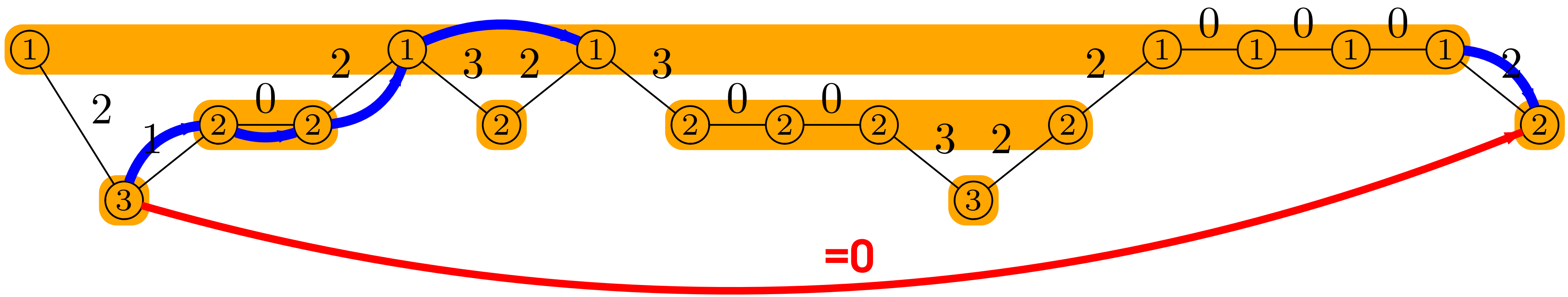

An elementary application

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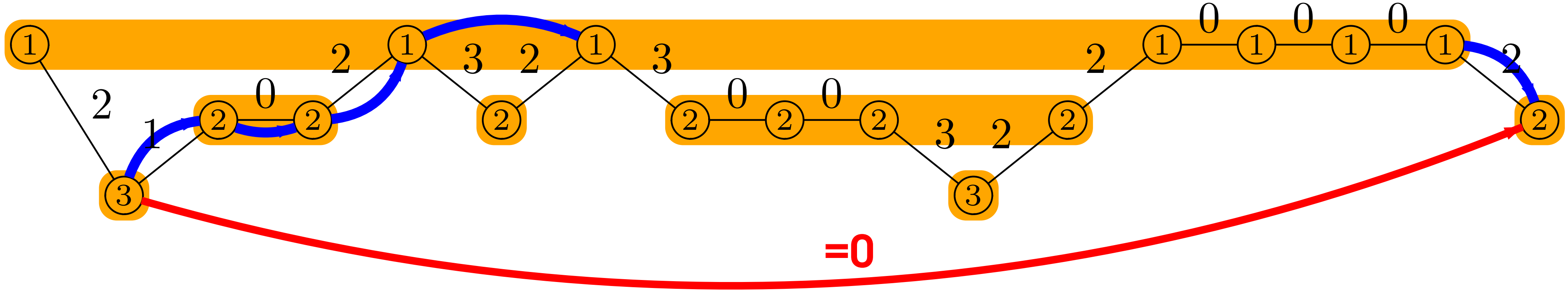

Conclusion: Computing $\sigma(x, y)$ for a given x, y is constant.

SOME ARGUMENTS OF THE PROOF

Situation 1: $(S,.)$ is a group
every element has an inverse

Situation 2: $ab = aa = a$, $ba = bb = b$
every product collapse to its first factor

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The proof uses a different argument in each situation.

Those arguments are combined in a proof for every finite semigroup (Green's relations).

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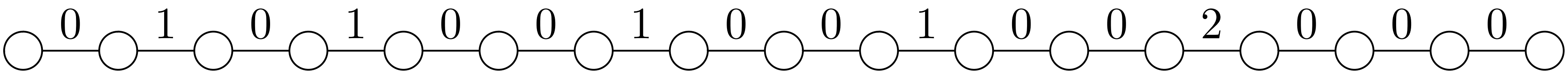
Group case. E.g., $(\mathbb{Z}/3\mathbb{Z}, +)$



Number S by n. Set $s(x) = n(\sigma(0, x))$, $s(0) = n(e)$.

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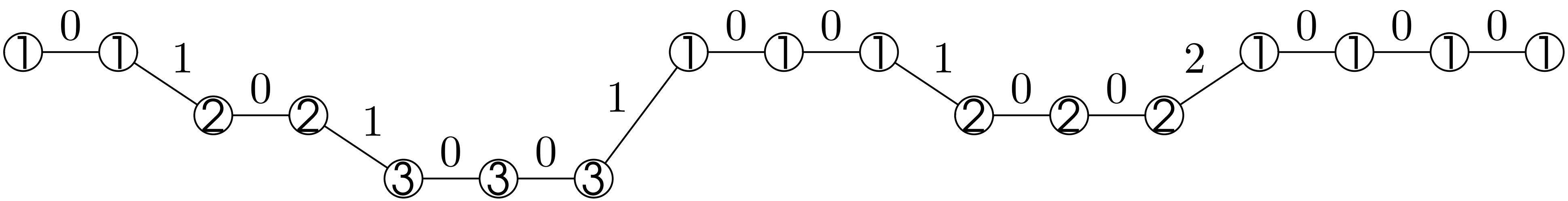

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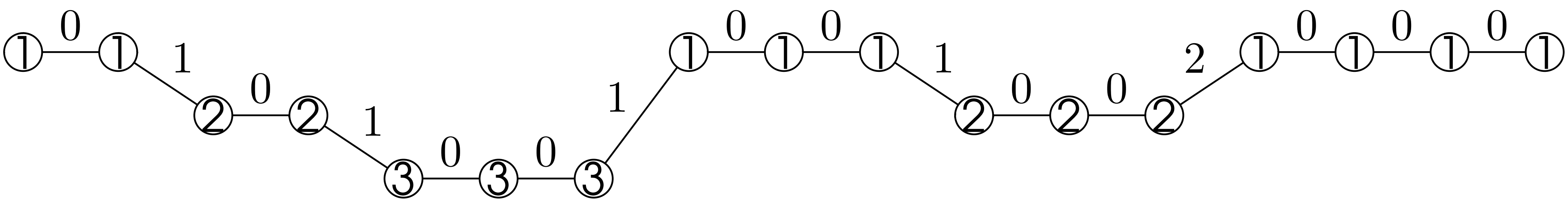

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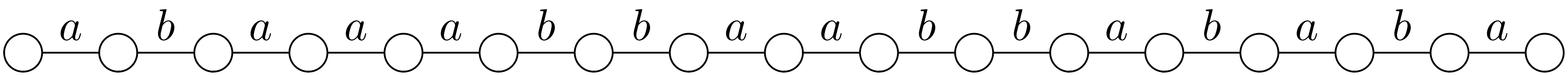

Number S by $n : S \to [1, N]$. Set $s(x) = n(\sigma(x - 1, x))$, $s(0) = 1$.

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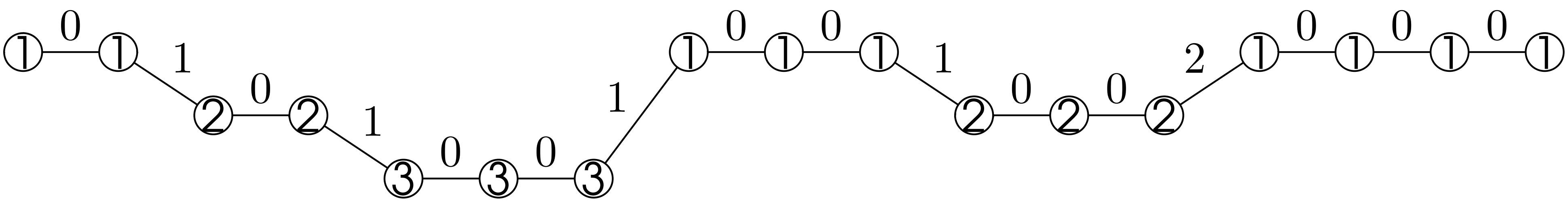

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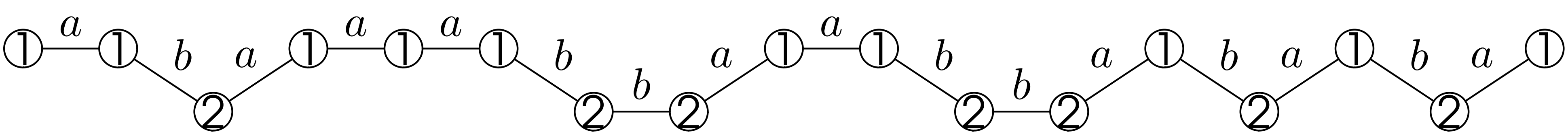

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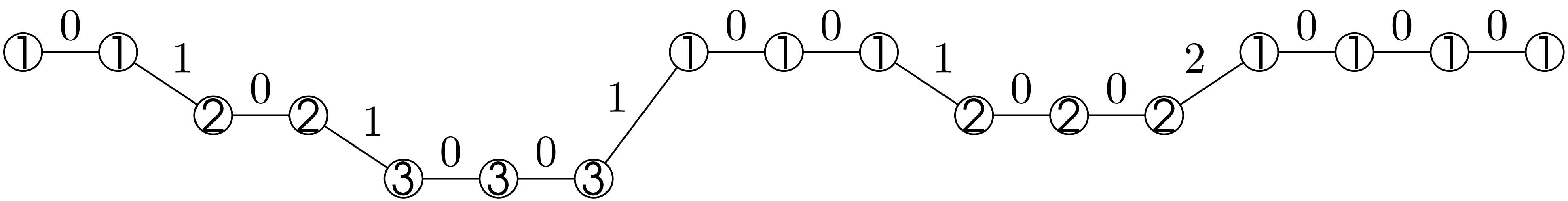

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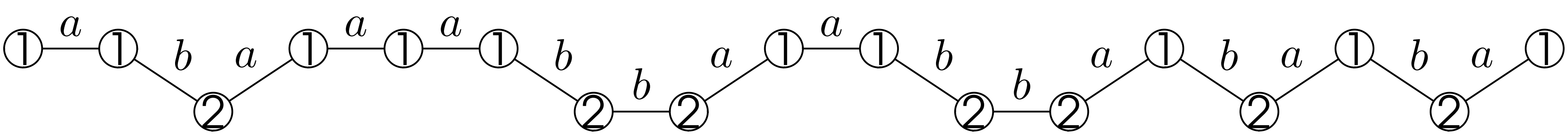

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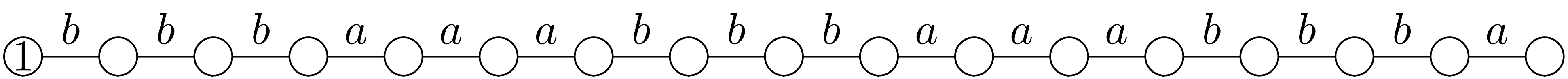

Fix a minimal remaining $\mathcal{J}$-class. Cut at the first occurence of this class. Proceed.

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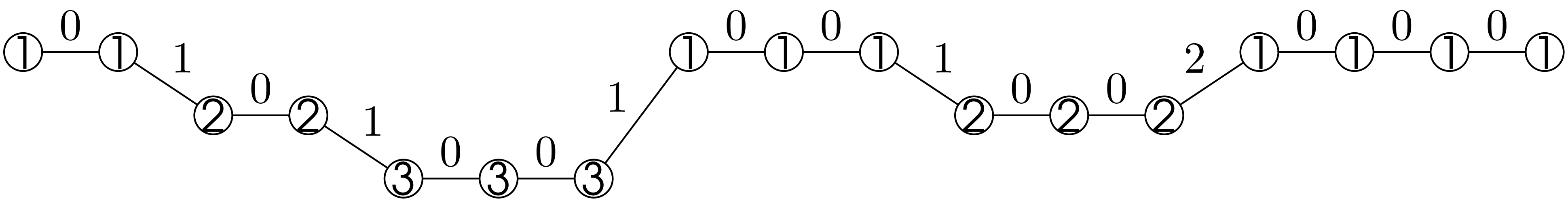

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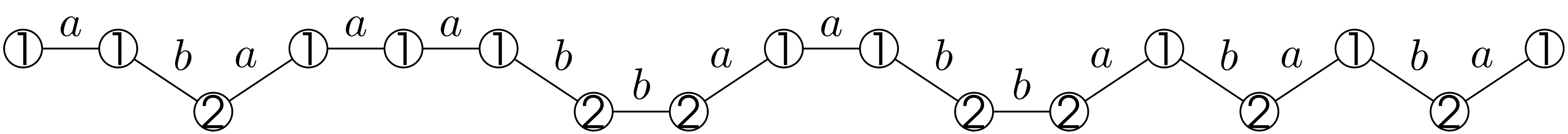

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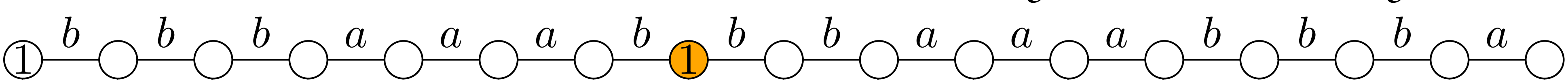

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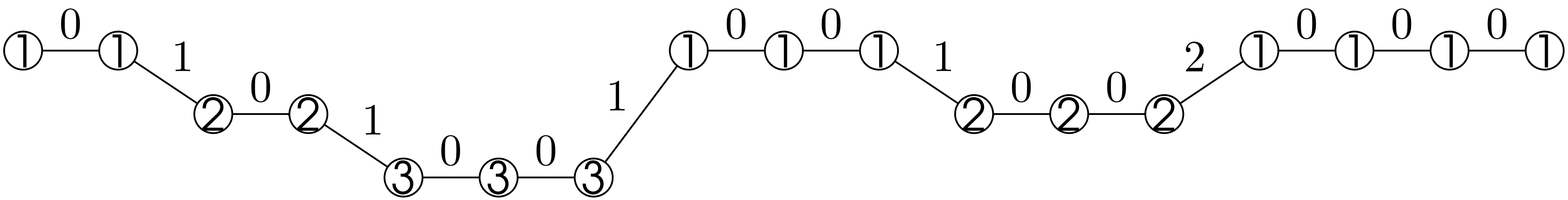

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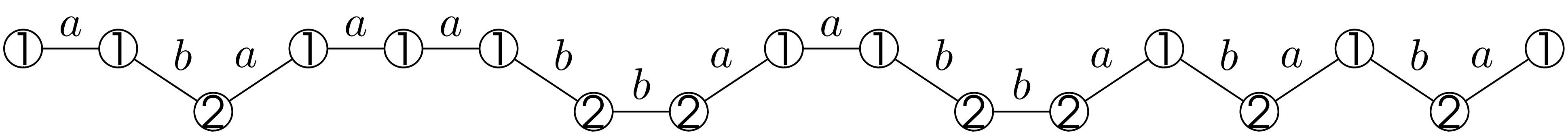

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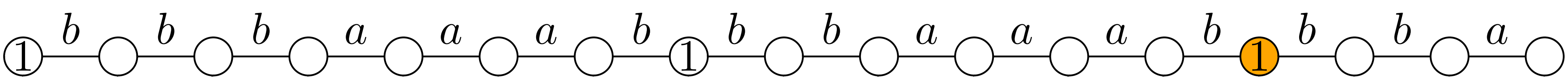

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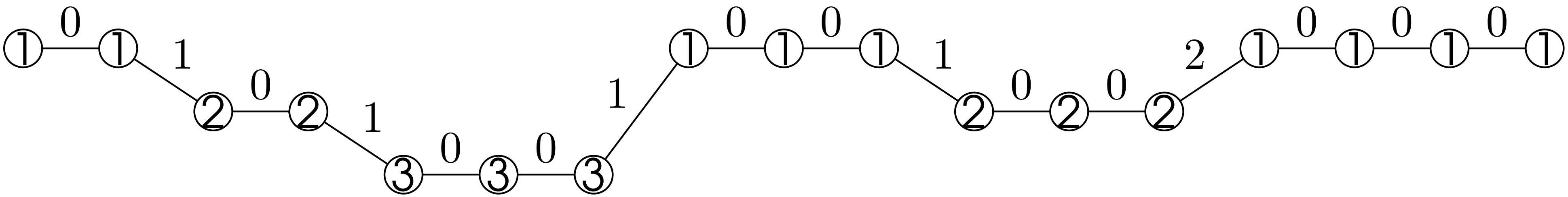

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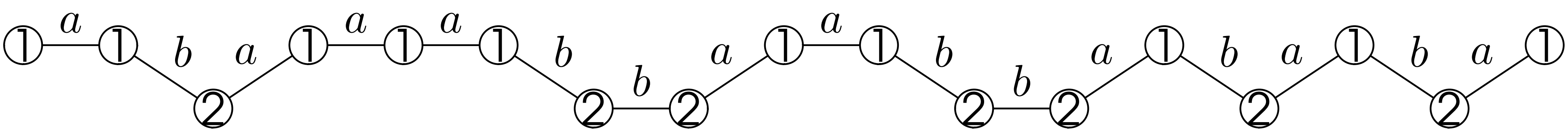
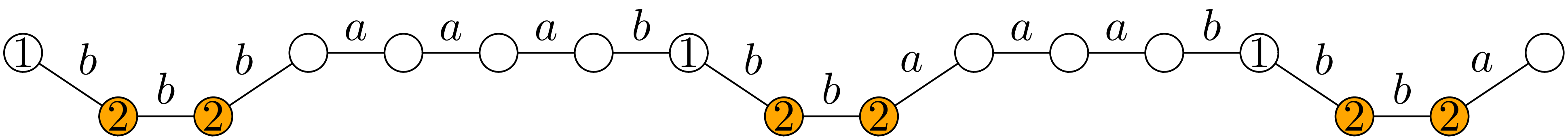


Number S by $n : S \to [1, N]$. Set $s(x) = n(\sigma(x-1, x))$, $s(0) = 1$.

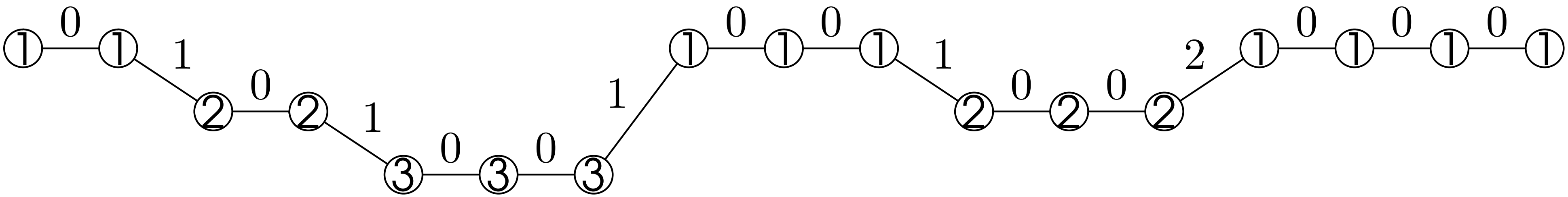
Case of different $\mathcal{J}$-classes. $c = {*c} = c{*} = ab \quad \leq_{\mathcal{J}} \quad b = bb = ba \quad \leq_{\mathcal{J}} \quad a = aa$



Fix a minimal remaining $\mathcal{J}$-class. Cut at the first occurence of this class. Proceed.

Some arguments of the proof

Group case. E.g., $(\mathbb{Z}/3\mathbb{Z}, +)$



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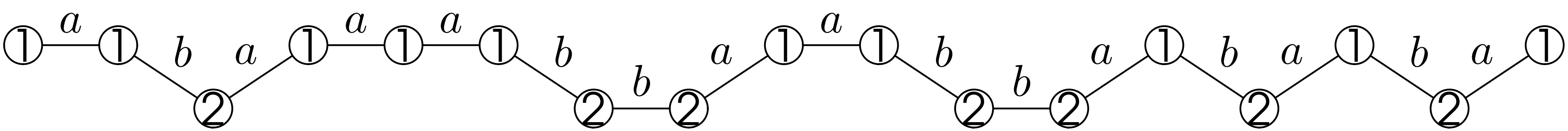

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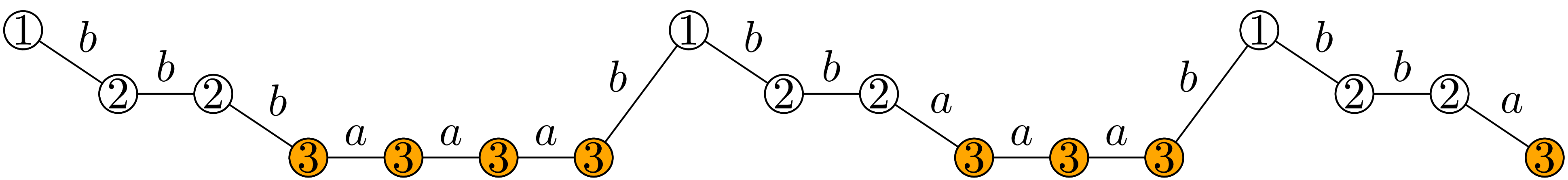

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Extension to infinite linear orderings

Th: Every additive labeling of a finite linear ordering by a finite semigroup S admits a Ramseyan split of height at most $|S|$.

Th: Every additive labeling of a complete linear ordering by a finite semigroup S admits a Ramseyan split of height at most $3|S|$.

Scattered linear orderings and automata

Buchi introduces automata accepting words indexed by ω.

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Contribution: A simpler proof using splits of complete linear orderings.

Related work

Other uses of Simon's factorisation theorem

- caracterisation of subfamilies of regular languages (Pin&Weil)
- limitedness of distance automata (Simon)
- extension of regularity over ω-words (Bojańczyk&C.)

Variants

- Deterministic variant, application to logic over trees (Icalp 07)