

Omega Regularity with Bounds

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GOAL: Consider another notion — more expressive — of regularity

EXTENDING ω -REGULAR EXPRESSIONS

Syntax of ω -regular expressions:

$$R = \emptyset \mid \varepsilon \mid a \mid R + R \mid R.R \mid R^*$$

$$O = R^\omega \mid R.O \mid O + O$$

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- B constrains the number of iteration to be bounded (B stands for 'bounded')

$(a^B b)^\omega$ there is an infinite number of b 's and,
the number of a 's between consecutive b 's is bounded

$$ababab \dots \in (a^B b)^\omega$$

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- B and S are variants of the Kleene-star $*$
- B constrains the number of iteration to be bounded (B stands for 'bounded')
- S constrains the number of iterations to tend toward the infinite (S stands for 'strictly unbounded')

$(a^S b)^\omega$ there is an infinite number of b 's and,
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- $U^\omega \mapsto \{u_1u_2 \dots : \vec{u} \in U\}$

OTHER EXAMPLES

$$((a^*b)^*a^Sb)^\omega$$

The languages with an infinite number of b 's such that the size of segments of a is not bounded.

E.g. $aba^2baba^3baba^4b \dots \in ((a^*b)^*a^Sb)^\omega$

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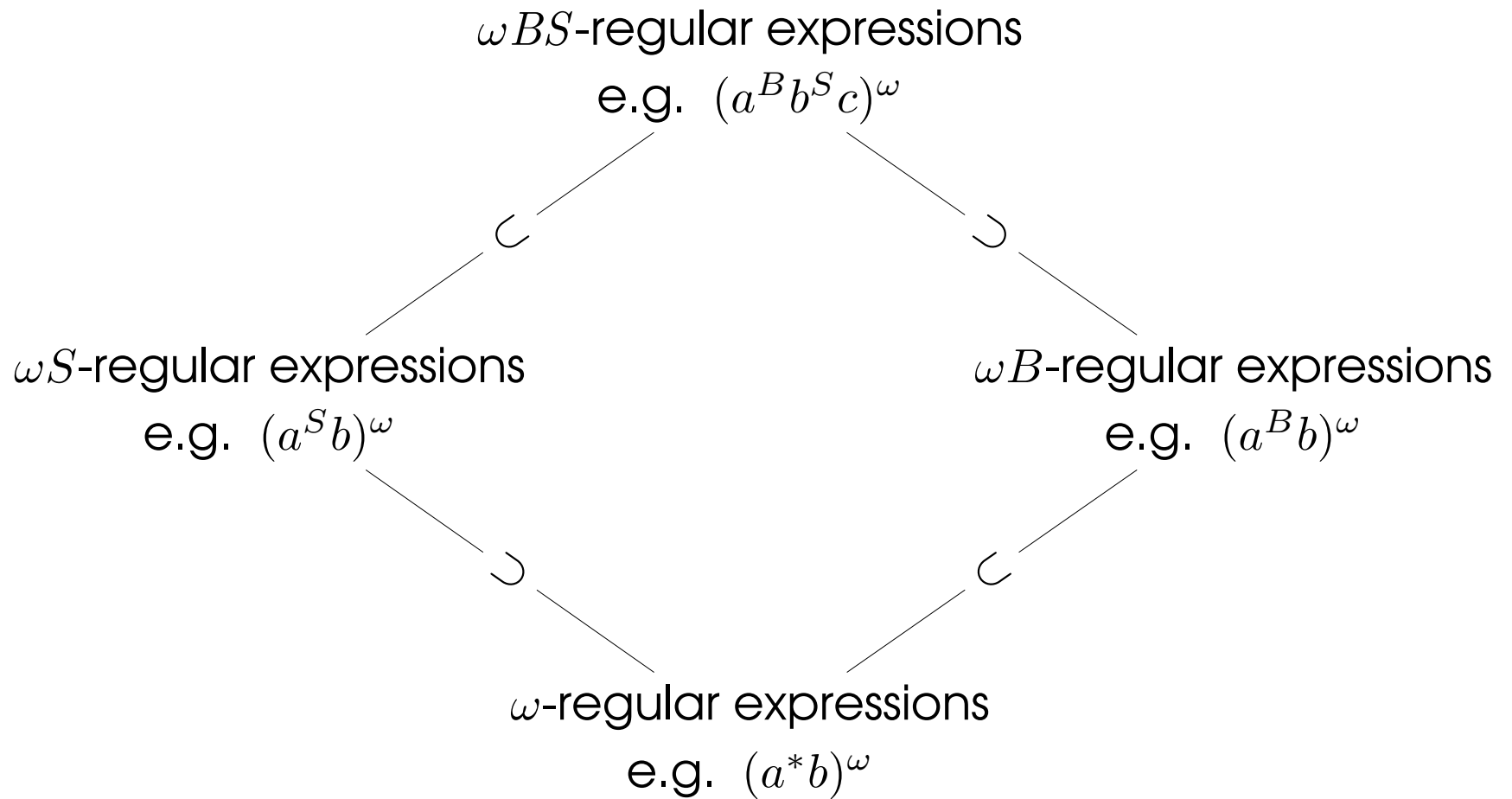
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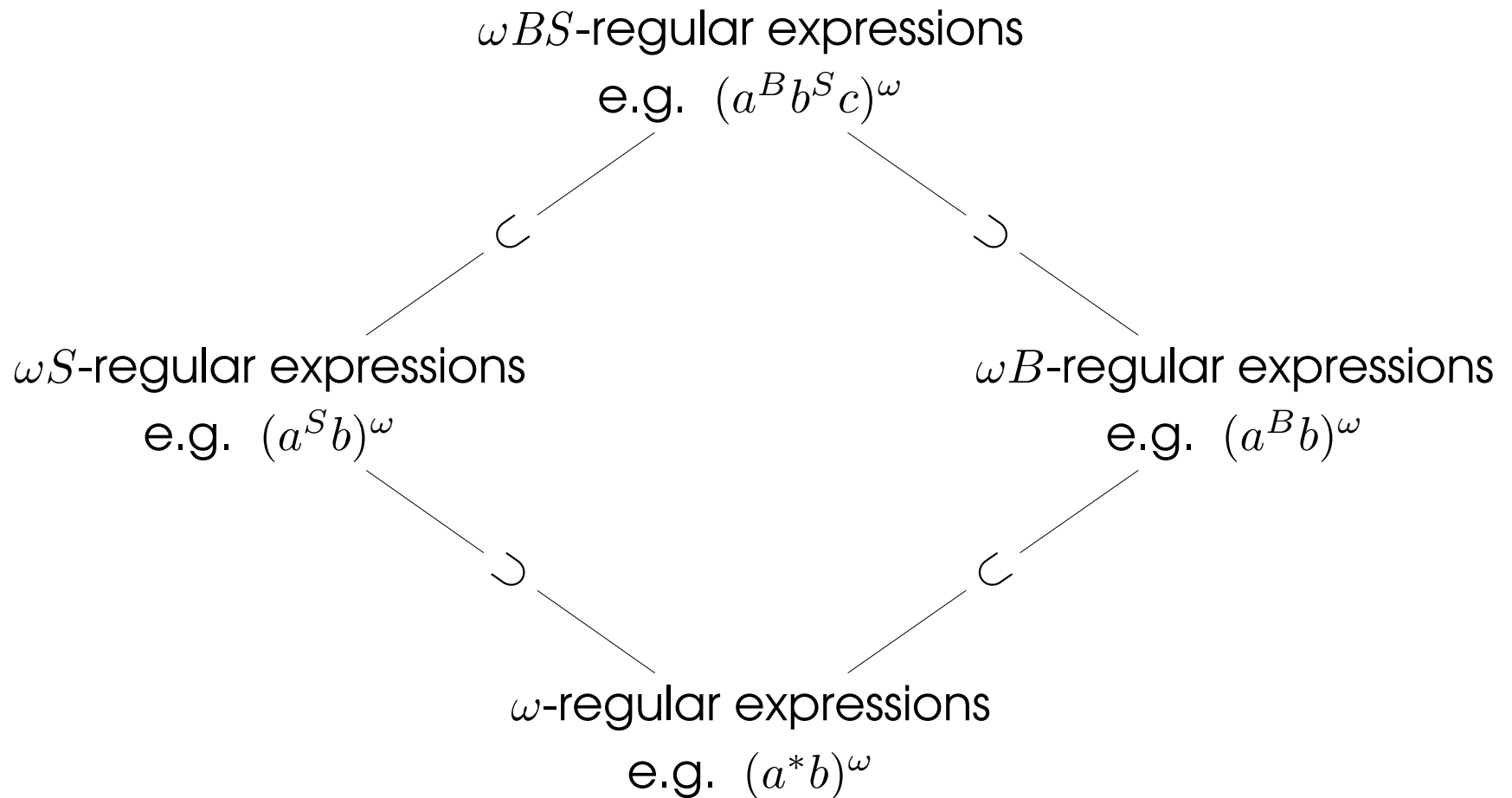
$$((a^Sb)^S a^S c)^\omega$$

There is an infinite number of c 's. The number of b 's between two c 's tends toward the infinite. The number of a 's between two b or c 's tends toward the infinite.

THE DIAMOND

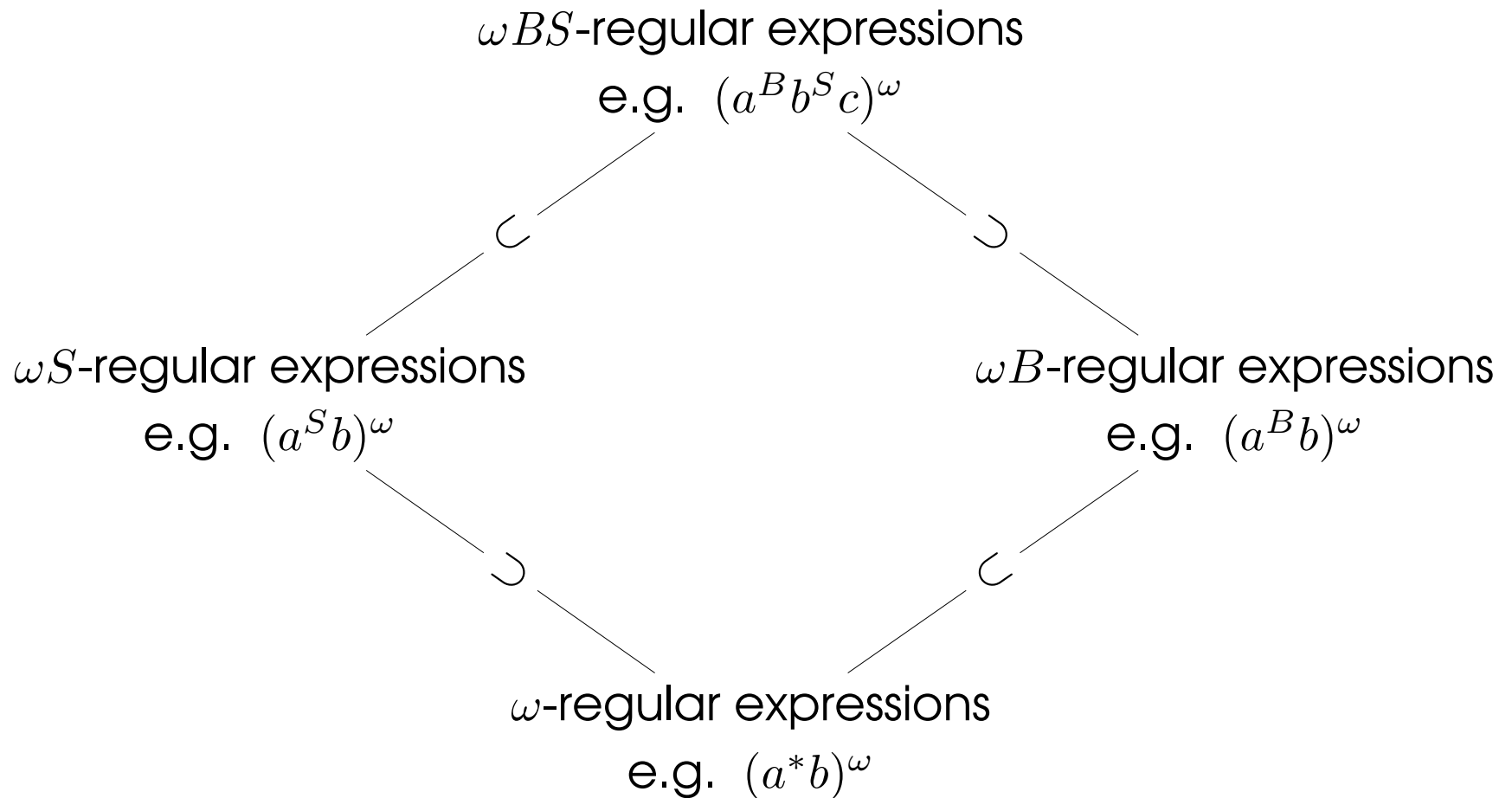


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PROP: The inclusions in the diamond are strict.

AUTOMATA

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Essentially: finite state automata with modified accepting condition (more expressive than Buchi).

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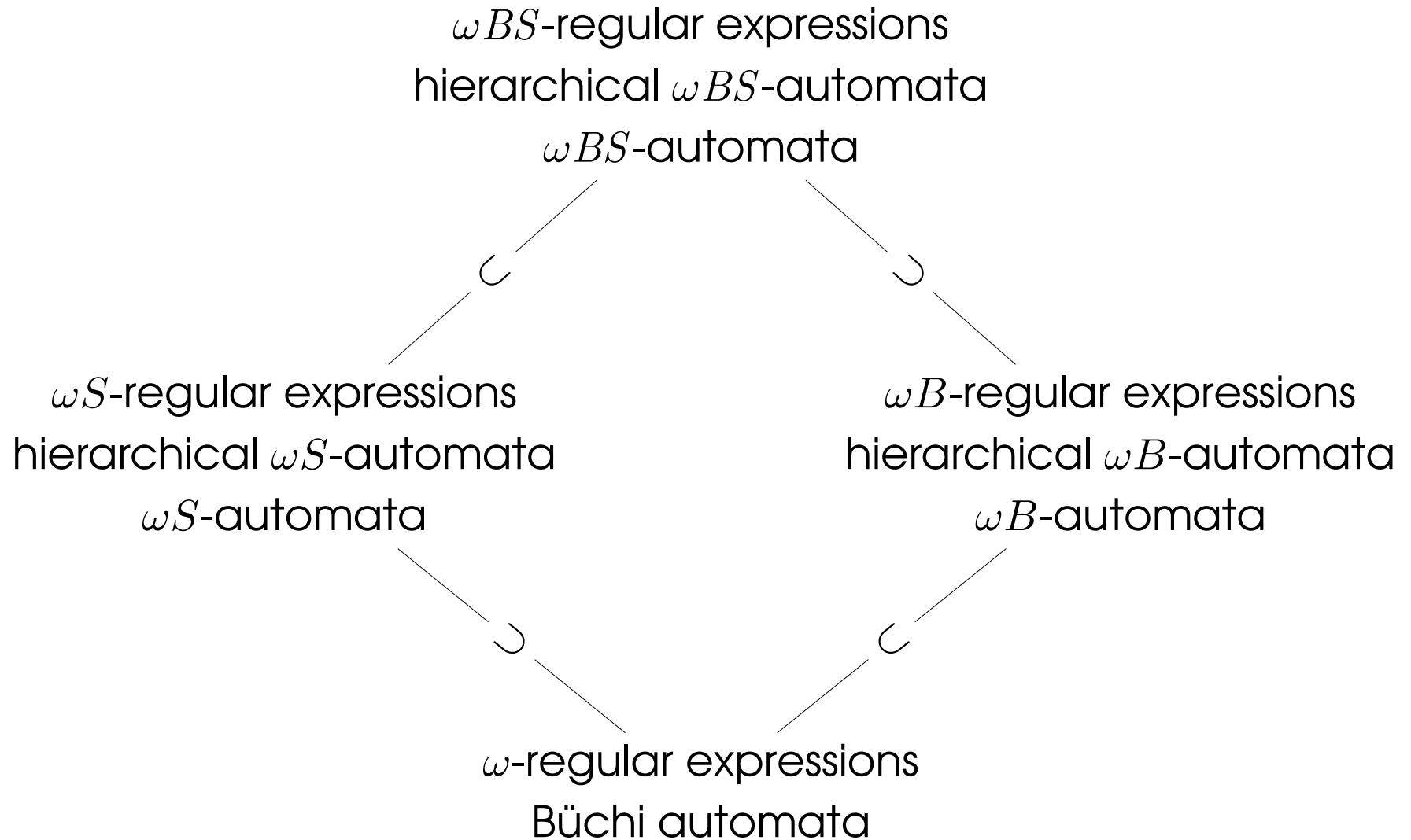
They come in two variants hierarchical or not.

THM: The following are equivalent:

- L is evaluation of an ωBS -regular expression
- L is accepted by an ωBS -automaton
- L is accepted by a hierarchical ωBS -automaton

And the same holds for ωB and ωS regular languages.

THE DIAMOND (2:AUTOMATA)



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Mp: The complement of $L = ((a^B + a^S)\#)^\omega$ is not ωBS -regular.

The language \bar{L} contains $a^{f(1)}ba^{f(2)}b \dots$ iff there exists infinitely many values appearing infinitely often in f .

E.g: $a^1b \ a^1ba^2b \ a^1ba^2ba^3b \ \dots \in \bar{L}$

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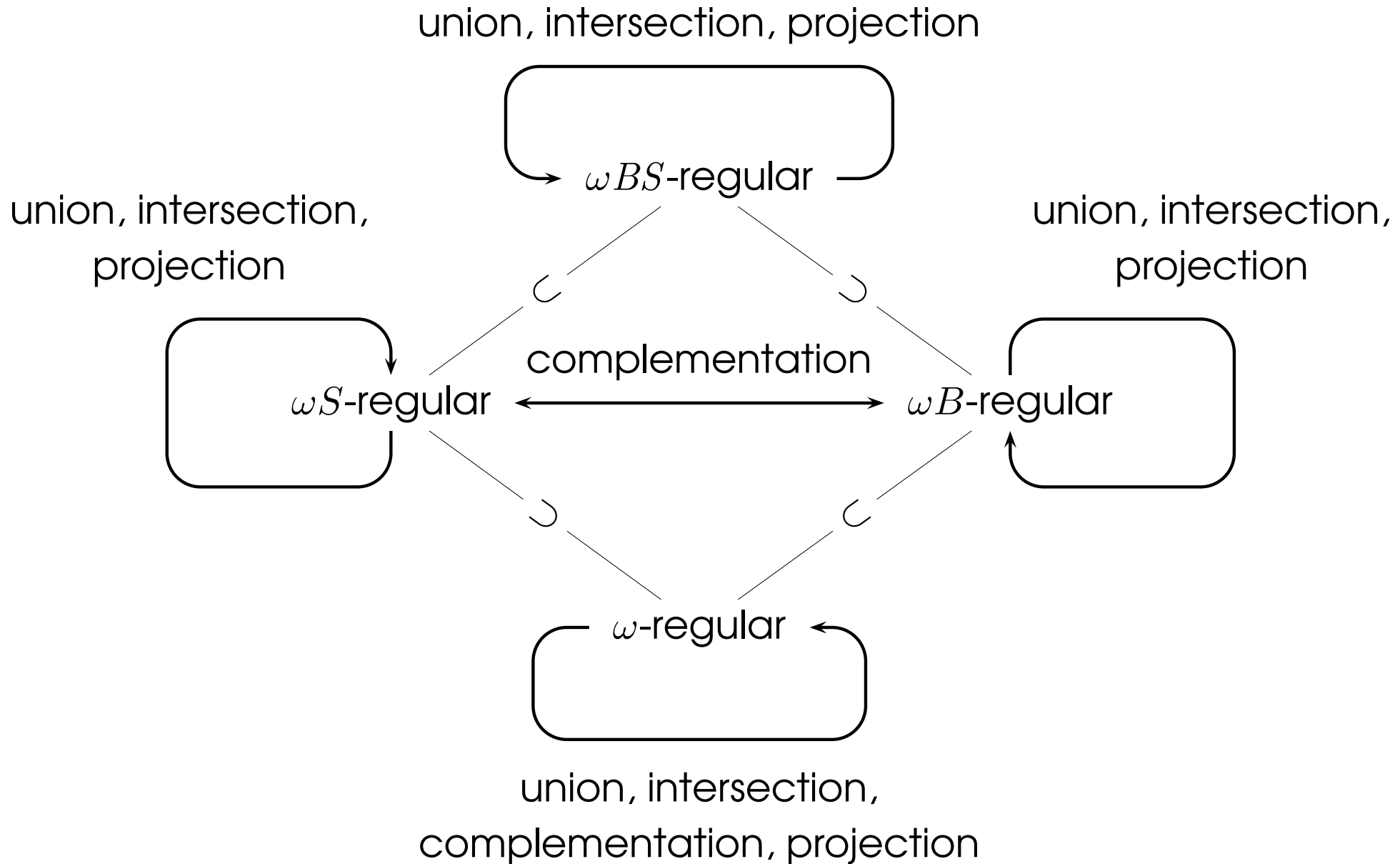
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THE DIAMOND (3:CLOSURE)



THE LOGIC MSOLB

The logic MSOL :

$$\phi = \phi \vee \phi \mid \phi \wedge \phi \mid \neg \phi \mid \exists X.\phi \mid \exists x.\phi \mid x \in X \mid x = S(y) \mid a(x)$$

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With $\mathbf{UX}.\phi \equiv \forall n. \exists X. (|X| > n) \wedge \phi$

“there exists a set as big as I want”

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QUESTION: Is SAT of MSOLB decidable over ω -words?

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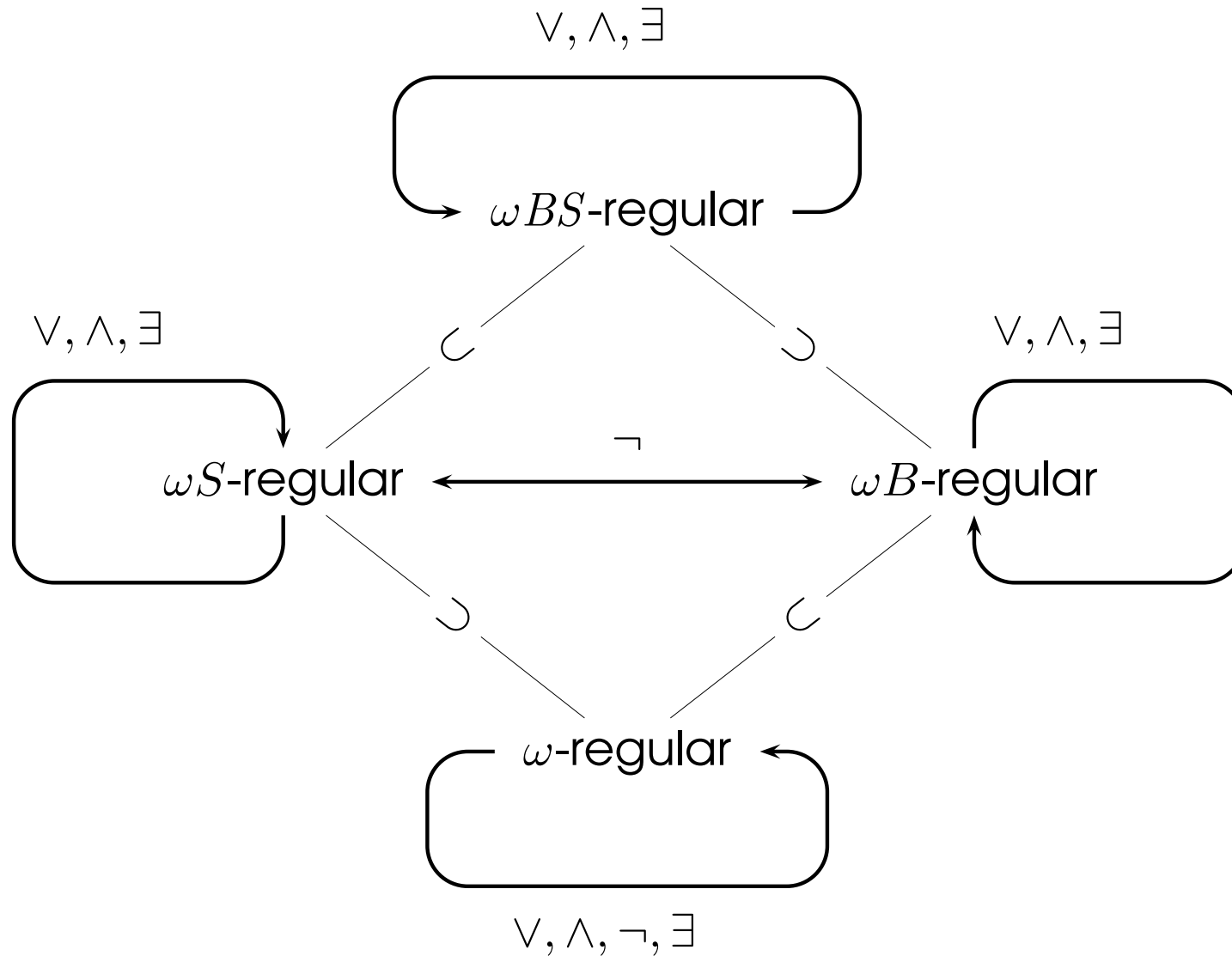
THM(Buchi): SAT of MSOL is decidable over infinite words.

Rq: Adding equality/comparison of cardinality to MSOL leads to undecidability of SAT (already for finite words).

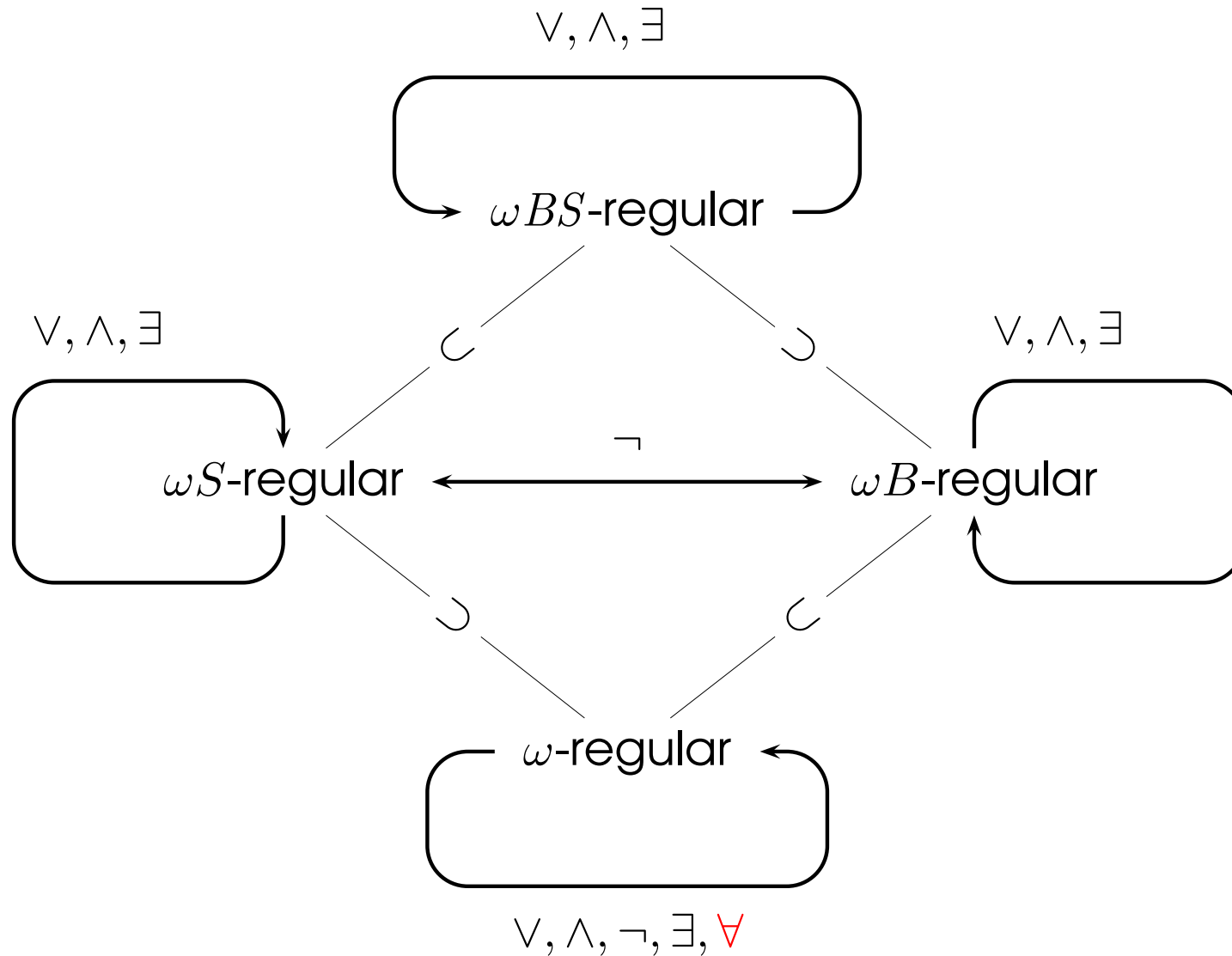
QUESTION: Is SAT of MSOLB decidable over ω -words? **The question is open.**

But ωBS -regularity provides a partial answer.

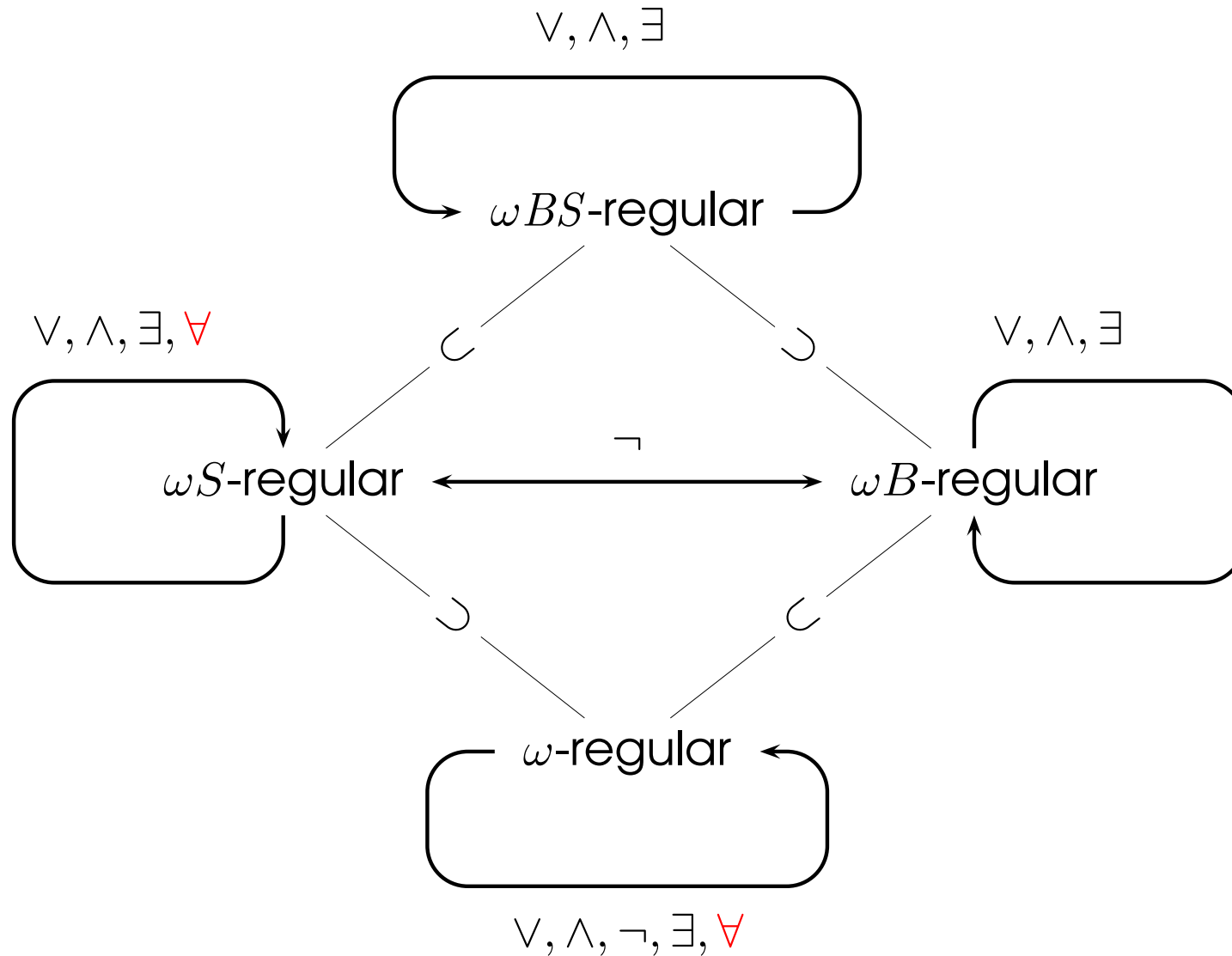
THE DIAMOND (3:LOGIC)



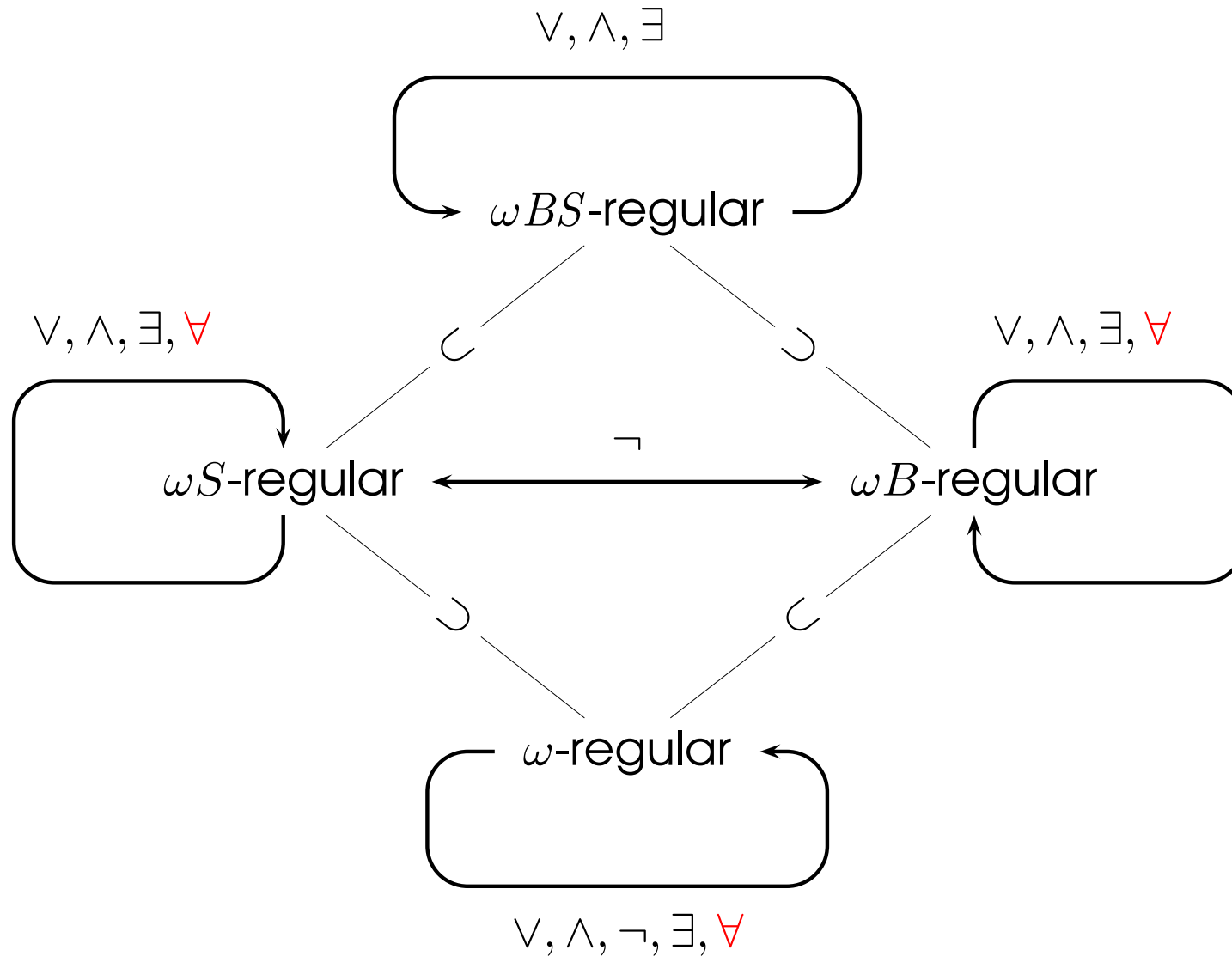
THE DIAMOND (3:LOGIC)



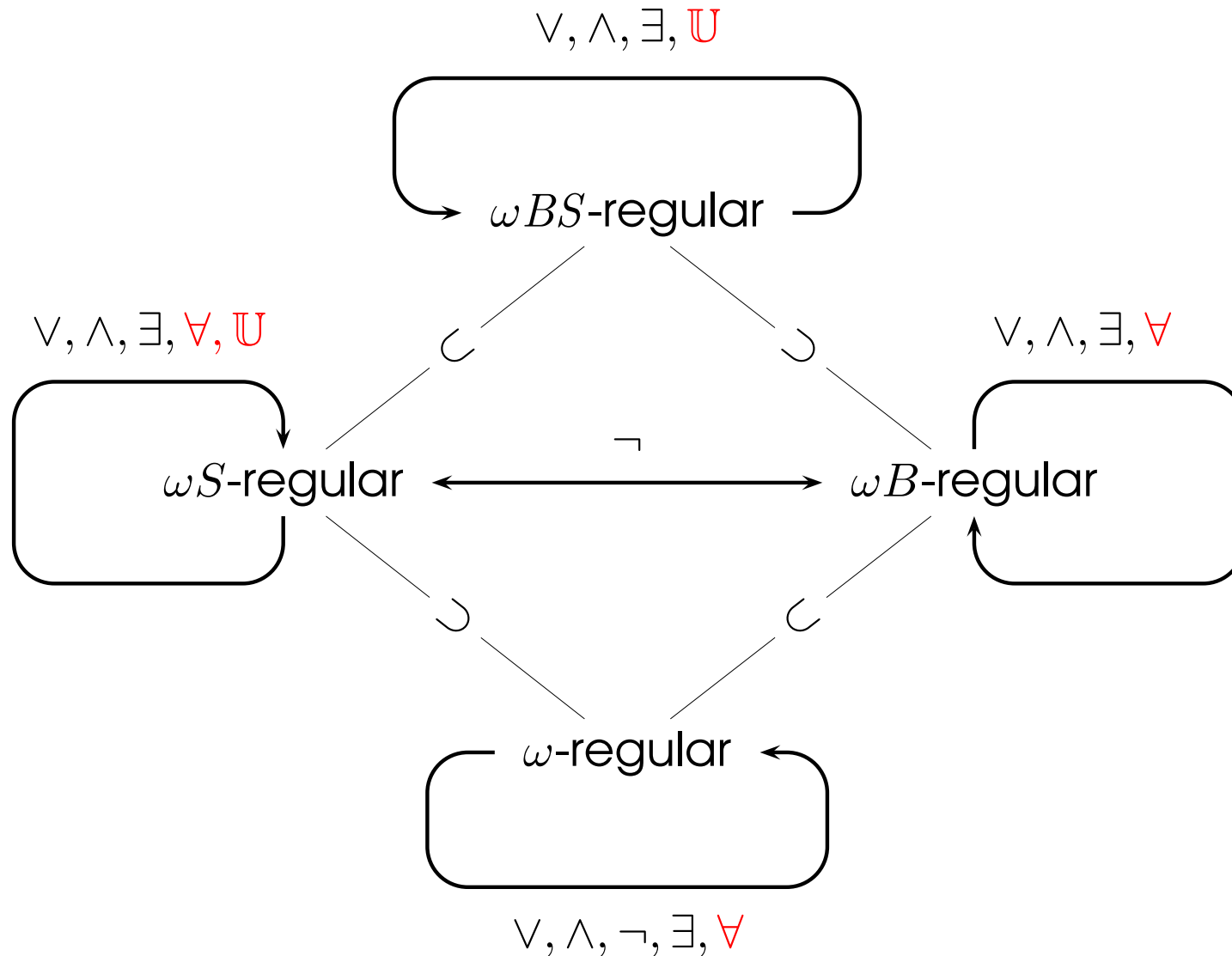
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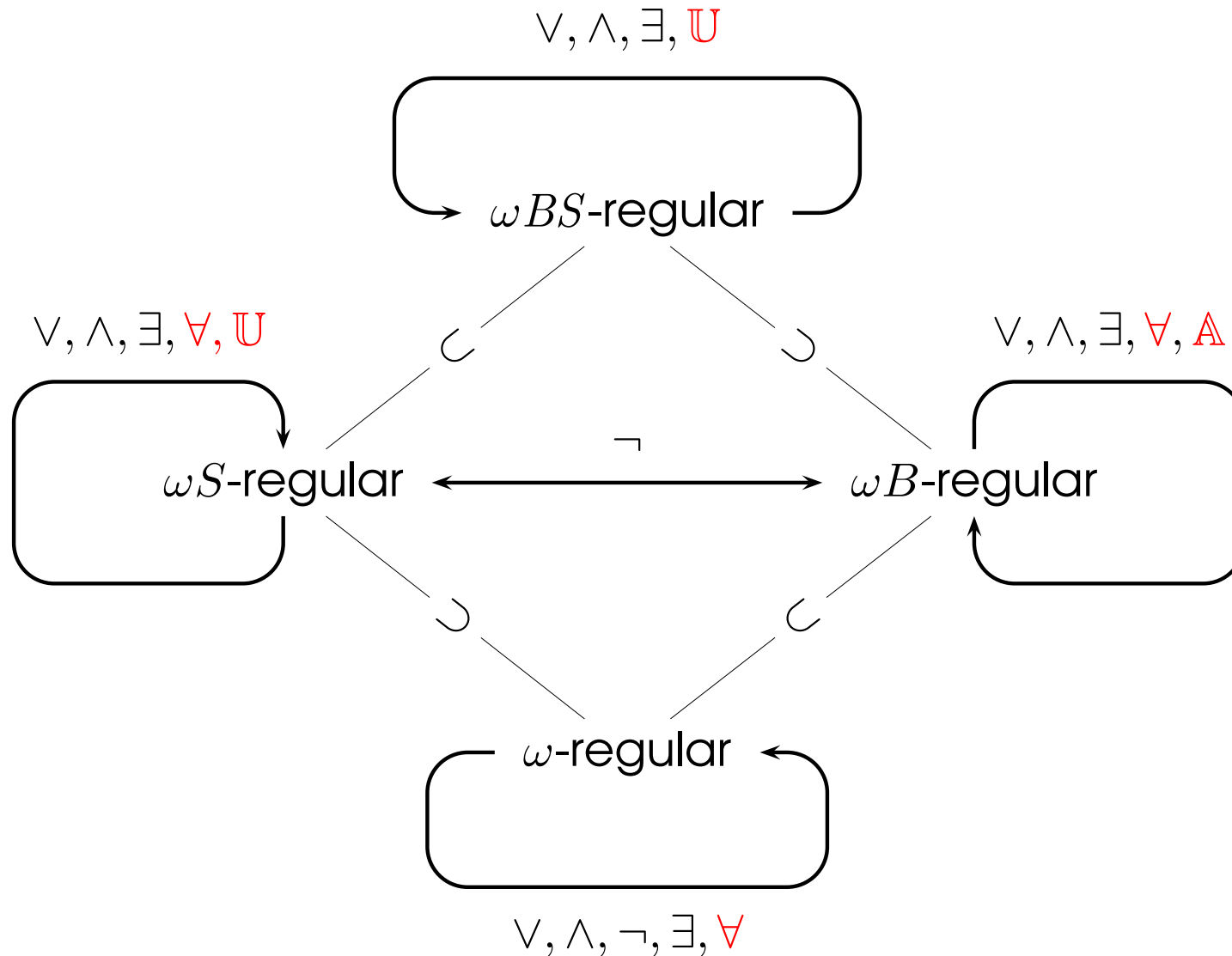


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PROP: ωS and ωBS -regular languages are closed under \cup .

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COR: MSOL is equivalent to ω -regular languages
MSOLB+ is equivalent to ωS -regular languages
MSOLB- is equivalent to ωB -regular languages
Boolean comb. of MSOLB+ are contained in ωBS -regular languages
 \Rightarrow **SAT is decidable for those fragments of MSOLB**

CONCLUSION

We have:

- Introduced an extension of ω -regular expressions.

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Two main open questions.

- Solve the full logic MSOLB over ω -words.
- Find equivalent class of languages over trees.