Positional determinacy over finite games

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Games on graphs of infinite duration

Arena: $\mathcal{A} = (V_E, V_A, E)$

with V_E is the set of positions belonging to Eva

V_A is the set of positions belonging to Adam

V_E and V_A are disjoints

$E \subseteq V \times \Sigma \times V$ a set of edjes

with $V = V_E \cup V_A$

Σ a finite alphabet fixed for the remaining of the talk

GAMES ON GRAPHS OF INFINITE DURATION

Arena: $\mathcal{A} = (V_E, V_A, E)$

Winning condition: $W \subseteq \Sigma^\omega$ with $\Sigma.W = W$ (prefix independence)

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with v_0 is an initial position in V

$\mathcal{A}$ is an arena

W is a winning condition

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Winner in a confrontation of strategies:
Given a strategy s_E for Eva and a strategy s_A for Adam.
Eva wins with s_E against s_A if the only play $(v_0, a_0, v_1)(v_1, a_1, v_2)\dots$
consistent with both s_E and s_A is such that $a_0 a_1 a_2 \cdots \in W$. Else Adam wins.

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Winner of a game/determinacy: Player Eva wins the game if there is a strategy s_E for Eva such that Eva wins with this strategy against all strategies of Adam.

The game is said determined if either Eva or Adam is winning it.

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A winning condition W is said positionaly determined if every game (of countable set of positions) is positionaly determined.

Parity condition: A winning condition W is a (generalized) parity condition if there exists $p : \Sigma \to \mathbb{N}$ such that

$$a_0 a_1 a_2 \cdots \in W \quad \text{iff} \quad \limsup_n p(a_n) \text{ is even.}$$

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TH:(Żielonka/Walukiewicz/C,Niwiński)
A winning condition is positionally determined iff it is a parity condition.

LIMITATIONS

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The left to right implication is not true anymore if

- W is not prefix independent (e.g. $W = a(a+b)^\omega$)
- labels are on vertices instead of edges (e.g. $(a+b)^*(ab)^\omega$)
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What about determinacy over finite games ?

Can we obtain a result of the form:

"A winning condition is positionaly determined over every finite games iff it is (to be completed) ?"

SOME WINNING CONDITIONS

Mean-payoff condition(Ehrenfeucht,Mycielski): W is a mean-payoff condition if there exists a monoid morphism $\rho : \Sigma^* \to \mathbb{R}$ such that

$$a_0 a_1 \cdots \in W \quad \text{iff} \quad \limsup_n \frac{\rho(a_0 a_1 \ldots a_n)}{n} > 0 \qquad \text{(we write } W = MPC(\rho)\text{)}$$

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Unboundedness condition: W is an unboundedness condition if there exists a monoid morphism $\rho : \Sigma^* \to \mathbb{R}$ such that

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TH: Both mean-payoff and unboundedness conditions are positionaly determined over finite games.

Furthermore for every finite arena $\mathcal{A}$ the same player wins the game $(v_0, \mathcal{A}, UC(\rho))$ and the game $(v_0, \mathcal{A}, MPC(\rho))$, and the positional winning strategies are the same in both games.

The two conditions are distinct but indistinguishable...

EQUIVALENCE BETWEEN CONDITIONS

Fact: Let W and W' be winning conditions positionaly determined over finite games.
Then the games $(v_0, \mathcal{A}, W)$ and $(v_0, \mathcal{A}, W')$ have the same winner for every finite arena $\mathcal{A}$ iff W and W' coincide on periodic words.
And in this case the positional winning strategies are the same.

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Equivalence between conditions

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Remark: Some equivalence classes contain an uncountable number of conditions.

Can we characterize the set of periodic words contained in a winning condition positionaly determined over finite games ?

Can we characterize positionaly determined winning condition up to equivalence?

PARTIAL ANSWER: PERMUTING CONDITIONS

Def: A winning condition is permuting if whenever $(uvw)^\omega$ is in W then $(uwv)^\omega$ is also in W.

Then a word u is completely characterized by its Parikh vector $p(u)$.
Let E be $\{p(u) \ : \ u^\omega \in W\}$.

Partial answer: permuting conditions

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Remark: For a condition W positionaly determined over finite games,
If $u^\omega \in W$ and $v^\omega \in W$ then $(uv)^\omega$. (The same holds for $\overline{W}$.)
Then E as well as $\overline{E}$ are closed under addition.

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From this remark we can conclude:
TH: Every permuting winning condition positionaly determined over finite games is equivalent to a generalized mean-payoff condition.
(The generalized mean-payoff conditions are positionaly determined over finite games.)

Def: A winning condition W is a chain mean-payoff condition if there exists monoid morphisms $\rho_1, \rho_2, \ldots, \rho_k : \Sigma^* \to \mathbb{R}$ such that

$$a_0 a_1 a_2 a_3 \cdots \in W \quad \text{iff for some } m, \begin{cases} \text{for } l < m & \limsup_n \frac{\rho_l(a_0 a_1 \ldots a_n)}{n} = 0 \\ \text{and} & \limsup_n \frac{\rho_m(a_0 a_1 \ldots a_n)}{n} > 0 \end{cases}$$

PROVING POSITIONAL DETERMINACY

TH(Gimbert,Żielonka): If every finite game with condition W where only one player plays is positionaly determined, then W is positionaly determined over every finite games.

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We need only to consider one-player games!

Strategy improvements

We assume games played only by Eva.
A strategy is then simply an infinite path, that we write as a word in $(V\Sigma)^*$.

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What should we assume on W for ensuring that the loop terminates?

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Let $V = V_E$ be $\{1, 2, 3, 4, 5, 6, 7\}$.

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Remark 1: The word appearing in each parenthesis — omitting inner parentheses — is an elementary cycle loosing for Eva.

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Let $V = V_E$ be $\{1, 2, 3, 4, 5, 6, 7\}$.

Step 0: $1\ a\ 2\ a\ 3\ a\ 4\ a\ 5\ a\ 5\ a\ 5\ a\ 6\ a\ 7\ a\ 7\ a\ 4\ a\ 5\ a\ 2\ a\ 3\ a\ 3\ a\ 3 \ldots$

Step 1: $1\ a\ 2\ a\ 3\ a\ 4\ a\ \quad 5\ a\ 5\ a\ 6\ a\ 7\ a\ 7\ a\ 4\ a\ 5\ a\ 2\ a\ 3\ a\ 3\ a\ 3 \ldots$

Step 2: $1\ a\ 2\ a\ 3\ a\ 4\ a\ \quad 5\ a\ 6\ a\ 7\ a\ 7\ a\ 4\ a\ 5\ a\ 2\ a\ 3\ a\ 3\ a\ 3 \ldots$

Step 3: $1\ a\ 2\ a\ 3\ a\ 4\ a\ \quad 5\ a\ 6\ a\ \quad 7\ a\ 4\ a\ 5\ a\ 2\ a\ 3\ a\ 3\ a\ 3 \ldots$

Step 4: $1\ a\ 2\ a\ 3\ a\ \quad 4\ a\ 5\ a\ 2\ a\ 3\ a\ 3\ a\ 3 \ldots$

Step 5: $1\ a\ \quad 2\ a\ 3\ a\ 3\ a\ 3 \ldots$

Step 6: $1\ a\ \quad 2\ a\ \quad 3\ a\ 3 \ldots$

Step 7: $1\ a\ \quad 2\ a\ \quad 3 \ldots$

We can construct a "parenthesed" version of the original strategy representing the successive improvement steps:

$$1\ a\ (2\ a\ 3\ a\ (4\ a\ (5\ a)(5\ a)\ 5\ a\ 6\ a\ (7\ a)\ 7\ a)\ 4\ a\ 5\ a)\ 2\ a\ (3\ a)(3\ a)\ 3 \ldots$$

Remark 1: The word appearing in each parenthesis — omitting inner parentheses — is an elementary cycle loosing for Eva.

Remark 2: Nesting depth of parentheses is bounded.

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Remark 1: The word appearing in each parenthesis — omitting inner parentheses — is an elementary cycle loosing for Eva.
Remark 2: Nesting depth of parentheses is bounded.

Conclusion: Assuming that words admitting a "parenthesisation" satisfying remarks 1 and 2 are loosing for Eva is sufficient for ensuring termination.

NECESSARY AND SUFFICIENT CONDITION

Weak interleaving:
W is weakly interleaving if for any $u_1 u_2 u_3 \cdots \in W$ and any $v^\omega \in W$ then $u_1 v u_2 v u_3 v \cdots \in W$.

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Lemma: Eva has positional winning strategies on every arena played only by her iff W is co-ultimately weakly interleaving.

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Lemma: Eva has positional winning strategies on every arena played only by her iff W is co-ultimately weakly interleaving.

TH: A winning condition is positionaly determined over finite games iff it is both ultimately weakly interleaving and co-ultimately weakly interleaving.

CONCLUSION

Main points

- Characterising wining condition over finite games is more difficult than over all games
- This “must” be done first up to equivalence over periodic words
- Under permuting hypothesis, chained unboundedness characterizes exactly positional determinacy over finite games up to equivalence
- We do not know of any winning condition positionaly determined over finite games which is not permuting
- “Parenthesisation” allows to characterize how should be the winning condition over non periodic words

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Questions

- Half positional determinacy ? Seems much more difficult. (partial results of Eryk Kopcynski)
- Why considering finite games or infinite games ? Are there usefull arenas of other shapes ?