HIGHER ORDER SMALLEST PARTS FUNCTIONS

This project will provide the foundation necessary for thesis work on a variety of questions at the intersection of combinatorics, basic hypergeometric series, and modular forms. This is currently an active area of research, thanks in large part to the developing notions of mock modular forms and harmonic Maass forms [4, 5], inspired by Ramanujan's mock theta functions. No background in number theory or hypergeometric series is required to get started.

Let $\overline{spt}(n)$ denote the sum, over all overpartitions λ of n, of the number of appearances of the smallest part of λ. For example, there are 14 overpartitions of 4,

$$4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ \overline{2}+2,\ 2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1,$$

giving $\overline{spt}(4) = 26$. This simple counting problem has surprising connections to number theory. For example, if we let $\overline{spt1}(n)$ denote the restriction of $\overline{spt}(n)$ to overpartitions whose smallest part is odd, then we have

$$\overline{spt1}(n) \equiv \left(\frac{n}{3}\right) H(n) \pmod 3,$$

where $\left(\frac{n}{3}\right)$ is the Legendre symbol and $H(n)$ denotes the Hurwitz class number of binary quadratic forms. This is a consequence of the fact that the generating function for $spt1(n)$ has the structure of a weight 3/2 *quasimock modular form* and is closely related to Zagier's weight 3/2 Eisenstein series [2].

Garvan [3] recently defined *higher order spt functions* for the case of ordinary partitions, i.e., overpartitions with no overlined parts. This was a non-trivial matter, depending crucially on being able to prove a certain conjectured inequality involving *rank and crank moments*. The full conjecture had resisted asymptotic methods for modular forms but finally succumbed to a certain iterative procedure for basic hypergeometric series known as the *Bailey chain*. Garvan went on to show that the higher *spt* functions for partitions satisfy a wealth of interesting congruence properties.

The goal of this project will be to apply Garvan's ideas in a more general setting, namely that of overpartitions and overpartition pairs. This will involve first formulating and proving appropriate inequalities involving generalized rank and crank moments [1]. Then a careful study of the generating functions will be required to achieve an elegant definition for the generalized higher order *spt* functions. Time permitting, it should be possible to employ facts about rank and crank moments to deduce arithmetic properties for the newly minted higher order *spt* functions.

References

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