2^{eme} Cours : Chordal Graphs
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Schedule

Comments on last course

A nice algorithmic problem coming from biology

An interesting problem to consider

Chordal graphs

Any graph solution for the rectangle problem ?

A good model

Start with the graph of the planar tiling and keep exactly 2 integers edges by rectangle.
This yields a graph (possibly with parallel edges) in which all vertices have even degrees except the corners.
Take a maximal path starting in one corner ....

An operation research problem

- Storage of products in fridges : each product is given with an interval of admissible temperatures.
  Find the minimum number of fridges needed to store all the products (a fridge is at a given temperature).
- A solution is given by computing a minimum partition into maximal cliques.
- Fortunately for an interval graph, this can be polynomially computed
- So knowing that a graph is an interval graph can help to solve a problem.

Common intervals

Let τ and σ be two permutations on $[1, n]$.
wlog τ is supposed to be the identity

2 permutations

$\tau =$ 1 2 3 4 5 6 7 8

$\sigma =$ 4 6 3 5 8 2 7 1

Definition

Two intervals $I, J \subseteq [1, n]$ are common intervals of τ and σ if $\tau(I) = \sigma(J)$ as subsets of $[1, n]$.

example

$\tau =$ 1 2 3 4 5 6 7 8

$\sigma =$ 4 6 3 5 8 2 7 1

The ordering of the elements may differ.

$[3, 6]$ and $[1, 4]$ are the unique non trivial maximal common intervals of τ and σ.

Problems

The size of the data is in $O(n)$.

1. Propose an algorithm which computes all non trivial maximal common intervals
2. Propose an algorithm which computes all common intervals
3. Same problems for k permutations
4. Same problems with some fixed number of errors

Maximal Common intervals

Maximal common interval to a set of permutations.
T. Uno and M. Yagura, linear time algorithm, Algorithmica 2000

Equivalent to find maximal **non trivial** common connected set when G_1 and G_2 are paths.

Equivalent to compute the modular decomposition of a permutation graph, when a representation is provided.
Several new $O(n)$ algorithms :
F. de Montgolfier, R. Mc Connell 2004
A. Bergeron, F. de Montgolfier, M. Raffinot, ESA 2005
B.M. Bui Xuan, MH, C. Paul, ISAAC 2005. . .

Definition

A graph is a chordal graph if every cycle of length ≥ 4 has a chord. Also called triangulated graphs, (chordaux in french)

First application : perfect phylogeny. Many NP-complete problems for general graphs are polynomial for chordal graphs.

Second application : graph theory. Treewidth (resp. pathwidth) are very important graph parameters that measure distance from a chordal graph (resp. interval graph).

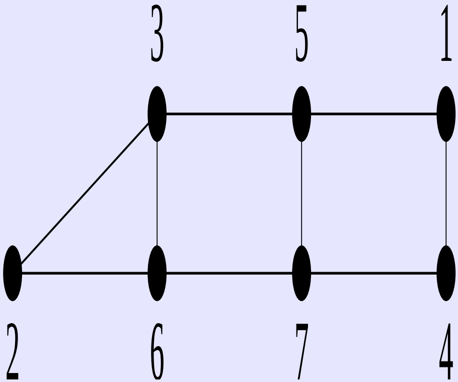
Basic facts

Chordal graphs are hereditary

Interval graphs are chordal

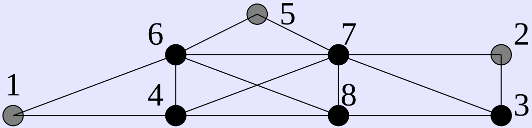
Lexicographic Breadth First Search (LexBFS)

Data: a graph $G = (V, E)$ and a start vertex s
Result: an ordering σ of V
Assign the label $\emptyset$ to all vertices
$label(s) \leftarrow \{n\}$
for $i \leftarrow n$ **à** 1 **do**
    Pick an unumbered vertex v with lexicographically largest label
    $\sigma(i) \leftarrow v$
    foreach *unnumbered vertex* w *adjacent to* v **do**
        $label(w) \leftarrow label(w).\{i\}$
    end
end


3
5
1
2
6
7
4

It is just a Breadth First Search with a strange tie-break rule !

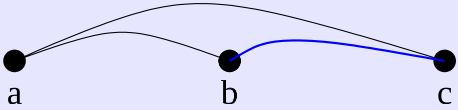
Chordal graph



A vertex is simplicial if its neighbourhood is a clique.

Simplicial elimination scheme

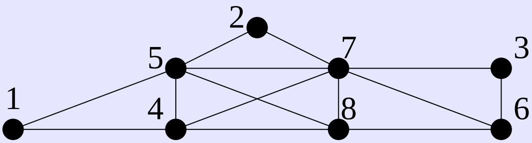
$\sigma = [x_1 \dots x_i \dots x_n]$ is a simplicial elimination scheme if x_i is simplicial in the subgraph $G_i = G[\{x_i \dots x_n\}]$



The reference for a graph algorithm theorem

LexBFS Characterization [Rose, Tarjan et Lueker 1976]

A graph is chordal G iff every LexBFS ordering of G provides a simplicial elimination scheme.



How can we prove such a theorem ?

1. A direct proof, finding the invariants ?
2. Find some structure of chordal graphs
3. Understand how LexBFS explores a chordal graph
4. We will consider the 3 viewpoints.

A characterization theorem for chordal graphs

Theorem

Dirac 1961, Fulkerson, Gross 1965, Gavril 1974, Rose, Tarjan, Lueker 1976.

(0) *G is chordal (every cycle of length ≥ 4 has a chord) .*

(i) *G has a simplicial elimination scheme*

(ii) *Every minimal separator is a clique*

Minimal Separators

A subset of vertices S is a minimal separator if G
if there exist $a, b \in G$ such that a and b are not connected in $G - S$.
and S is minimal for inclusion with this property .