Exam: MPRI Graph Algorithms

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1 Algorithms

1. Let us consider the following elimination ordering in which at each step a vertex of minimum degree (in the remaining graph) is chosen.

   Show that if the graph is chordal, the last eliminated vertices yield a clique of maximum size. Does this also work for cographs ?
2. Let us define the diameter of an undirected graph $G = (V, E)$, denoted by $diam(G)$ as the maximum over all shortest paths between vertices of V.

   i.e. $diam(G) = max_{x,y \in V}\{d(x, y)\}$.

   Design an algorithm to compute the diameter of a graph, and analyze its complexity.
3. Let us consider this algorithm :

Data: A graph $G = (V, E)$
Result: u, v two vertices
Choose a vertex $w \in V$
$u \leftarrow BFS(w)$
$v \leftarrow BFS(u)$

Where BFS stands for Breadth First Search.

Show that if G is a tree, $diam(G) = d(u, v)$

4. * If G is a chordal graph, what is the behavior of the previous algorithm ?
5. This definition can be extended to any subset $A \subseteq V$ and let us denote by $diam(A) = max_{x,y \in A}\{d_G(x, y)\}$

   By convention $diam(\emptyset) = 0$.

   We recall that a submodular function f must satisfy for every $A, B \subseteq V$:

   $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

   Prove or disprove that the diameter is a submodular function.

2 Tree-decomposition

Let us recall that a graph $G = (V, E)$ has a tree-decomposition $D = (S, T)$ if $S = \{S_1, S_2, \ldots, S_h\}$ is a collection of subsets of V, called bags, T a tree whose vertices are elements of S such that :

0 The union of elements in S is V

i $\forall e \in E$, $\exists i \in I$ with $e \in G(S_i)$.

ii $\forall x \in V$, the elements of S containing x form a subtree of T.

1. Give a complete proof of the following fact :
   Let S_1S_2 be an edge of T (joining the two bags S_1 and S_2), let T_1 and T_2 be the subtrees of T obtained by removing the edge S_1S_2. Then, $I = S_1 \cap S_2$ separates (i.e. is a separator in G) vertices of T_1-I from T_2-I.

3 Treelength

Let us consider a new graph parameter, denoted by $treelength(G)$ and defined as follows :

$Treelength(G) = min_{over\ all\ D}\{max_{S\ bag\ of D}\{diam(S)\}\}$

In other words, for treewidth one measures the maximum size of a bag, as for treelength one measures the maximum diameter of a bag.

1. Show that : $Treelength(G) = 1$ iff G is chordal.
2. Show that if G is a cograph then $Treelength(G) \leq 2$.
3. * Same question for distance hereditary graphs.
4. Compute the Treelength of C_n the cycle of length n.
5. * Compute the Treelength of the grid $G_{n,m}$
6. Show that Treewidth and Treelength are incomparable parameters. i.e. that there exist two graphs G, G' with $Treewidth(G) < Treelength(G)$ and $Treewidth(G') > Treelength(G')$.
7. When a graph is not a prime graph (with respect to modular decomposition) can we simplify the computation of $Treewidth(G)$ (resp. $Cliquewidth(G)$, $Treelength(G)$) using the modular decomposition tree ?
8. ** Does there exist a duality theorem for Treelength (analogous to the Bramble number for Treewidth).