Master Parisien de Recherche en Informatique

Theory of Computations

Assignment 6 — Handed out October 31, 2011

Exercise 1.

Show that a set A is Π^0_2 if and only if there exists a total computable function $F : \mathbb{N}^2 \to \{0,1\}$ such that for all n, n is in A if and only if $F(n,k) = 1$ for infinitely many k.

Exercise 2. Let $A \subseteq \mathbb{N}$ be some set. Show that the set $X = \{e \mid A \subseteq W_e\}$ is $\Pi^0_2[A]$ and computes A. For each of the following, give an example of set A making the equivalence hold: $X \equiv_T A$, $X \equiv_T A'$, and $X \equiv_T A''$.

Exercise 3.

Consider the following set

$$\mathsf{Ext} = \{e \mid \phi_e \text{ has a total computable extension}\}.$$

(by total computable extension of ϕ_e we mean a total computable function which coincides with ϕ_e on the domain of ϕ_e).

(a) Show that there exists a partial computable function which has no total computable extension. *Hint: one can consider for example the function* $n \mapsto \min\{t \mid n \in \emptyset'[t]\}$.

(b) Show that a partial computable function whose domain is computable has a total computable extension.

(c) Show that Ext is Σ^0_3.

(d*) Using the movable marker technique, show that Ext is Σ^0_3 many-one complete. *Hint: Proceed as in the proof of the completeness of* Cof *with numbered markers. We build a partial computable function f which will have no computable extension if and only if no marker moves infinitely often. The role of marker number i is to try to ensure that ϕ_i does not extend f. Originally all markers are "active". When the marker number i is placed on a position k and is active, the computation of $\phi_i(k)$ is run, and if it halts when the marker i is still at k, then define $f(k) = \phi_i(k) + 1$, declare the marker i to be "inactive" forever and move that marker (inactive markers still move). If a position k has no marker on it at some point and f is not yet defined at k, then set $f(k) = 42$... Fill the gaps and show that this construction works (note that if a marker moves infinitely often, then the constructed function will be defined on almost all n).*

Exercise 4.

Using the previous exercise (whether you solved it or not), show that the set

$$\mathsf{Comp} = \{e \mid W_e \text{ is computable}\}$$

is Σ^0_3 many-one complete. To see that it is many-one complete, it suffices to analyze the construction of exercise 3. Given a Σ^0_3 set A, exercise 3 shows how to, given n as a parameter, construct a partial computable function f such that

- If $n \in A$, f has co-finite domain, hence its domain is computable
- If $n \notin A$, f cannot be extended by any total computable function, hence by exercise 3 (b), its domain is not computable

Therefore, calling $j(n)$ the index of the function f, we have $n \in A \Leftrightarrow j(n) \in \mathsf{Comp}$. Therefore any Σ_3^0 set A is many-one reducible to Comp.