RAMANUJAN-TYPE PARTIAL THETA IDENTITIES AND CONJUGATE BAILEY PAIRS

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Abstract. Residual identities of Ramanujan-type partial theta identities are tailor-made for producing conjugate Bailey pairs. This is carried out for partial theta identities in Ramanujan's lost notebook, a number of the partial theta identities of Warnaar, and for some new ones as well.

1. Introduction

In his lost notebook, Ramanujan recorded a number of partial theta identities like the following [5, Entry 6.6.1]:

$$\begin{aligned}\sum_{n\geq 0}\frac{(q^{n+1})_n q^n}{(-a)_{n+1}(-q/a)_n} &= \sum_{n\geq 0}(-a)^n q^{n^2+n} \\ &\quad - \frac{a}{(-a,-q/a)_\infty}\sum_{n\geq 0}(-1)^n a^{3n} q^{3n^2+2n}(1+aq^{2n+1}).\end{aligned} \tag{1.1}$$

Here we have employed the usual basic hypergeometric series notation,

$$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$$

and

$$(a_1,a_2,\ldots,a_j)_n := (a_1,a_2,\ldots,a_j;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n.$$

The reason for applying the term *partial theta* to an identity like (1.1) is clear when one recalls the classical Jacobi theta function,

$$\sum_{n\in\mathbb{Z}} z^n q^{\binom{n}{2}} = (-z,-q/z,q)_\infty.$$

Ramanujan's partial theta identities were first proven by Andrews [1]. Later Warnaar [23] placed these identities in the context of Bailey pairs, established new examples and showed how one could generate many more.

A Bailey pair relative to (a,q) is a pair of sequences $(\alpha_n,\beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{r=0}^{n}\frac{\alpha_r}{(aq)_{n+r}(q)_{n-r}}. \tag{1.2}$$

A conjugate Bailey pair relative to (a,q) is a pair of sequences $(\delta_n,\gamma_n)_{n\geq 0}$ satisfying

$$\gamma_n = \sum_{r=n}^{\infty}\frac{\delta_r}{(aq)_{r+n}(q)_{r-n}}. \tag{1.3}$$

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Given a Bailey pair and a conjugate Bailey pair, a simple interchange of sums yields [9]

$$\sum_{n\geq 0}\beta_n\delta_n = \sum_{n\geq 0}\alpha_n\gamma_n. \tag{1.4}$$

In this paper we discuss how Ramanujan-type partial theta identities give rise, through their *residual identities*, to conjugate Bailey pairs. We treat six partial theta identities of Ramanujan, five identities stated by Warnaar, and six more identities we calculated using Warnaar's methods. These examples should be sufficient to illustrate the idea. The results are collected in the following theorem. Instead of stating the conjugate Bailey pair, we state the identity corresponding to (1.4).

Theorem 1.1. (1) *If (α_n,β_n) is a Bailey pair relative to (a,q), then*

$$\sum_{n\geq 0} q^n\beta_n = \frac{1}{(aq,q)_\infty}\sum_{r,n\geq 0}(-a)^n q^{\binom{n+1}{2}+(2n+1)r}\alpha_r. \tag{1.5}$$

(2) *If (α_n,β_n) is a Bailey pair relative to (a^2,q), then*

$$\sum_{n\geq 0}(a)_n q^n\beta_n = \frac{(a)_\infty}{(a^2q,q)_\infty}\sum_{r,n\geq 0}\frac{(-1)^n a^{3n}q^{n(3n+1)/2+3nr+r}(1-a^2q^{2n+2r+1})}{1-aq^r}\alpha_r. \tag{1.6}$$

(3) *If (α_n,β_n) is a Bailey pair relative to (a^2q^2,q^2), then*

$$\sum_{n\geq 0}(aq)_{2n}q^{2n}\beta_n = \frac{(aq)_\infty}{(a^2q^4,q^2;q^2)_\infty}\sum_{r,n\geq 0}\frac{q^{\binom{n+1}{2}+2nr+2r+n}a^n}{1-aq^{2r+1}}\alpha_r. \tag{1.7}$$

(4) *If (α_n,β_n) is a Bailey pair relative to (a^2q,q), then*

$$\sum_{n\geq 0}(aq)_n q^n\beta_n = \frac{(aq)_\infty}{(a^2q^2,q)_\infty}\sum_{r,n\geq 0}(-a)^{3n}q^{n(3n+5)/2+3nr+r}(1+aq^{n+r+1})\alpha_r. \tag{1.8}$$

(5) *If (α_n,β_n) is a Bailey pair relative to (a,q), then*

$$\sum_{n\geq 0}(aq;q^2)_n q^n\beta_n = \frac{1}{(aq^2;q^2)_\infty(q)_\infty}\sum_{r,n\geq 0}(-a)^n q^{n^2+2rn+r+n}\alpha_r. \tag{1.9}$$

(6) *If (α_n,β_n) is a Bailey pair relative to (a^2,q), then*

$$\sum_{n\geq 0}\frac{(a^2q)_{2n}q^n}{(aq)_n}\beta_n = \frac{1}{(aq,q)_\infty}\sum_{r,n\geq 0}a^{3n}q^{3n^2+2n+3rn+r}(1-aq^{2n+r+1})\alpha_r. \tag{1.10}$$

(7) *If (α_n,β_n) is a Bailey pair relative to (a^2,q), then*

$$\sum_{n\geq 0}\frac{(a^2q)_{2n}q^n}{(aq,aq^2)_n}\beta_n = \frac{1}{(1-q)(q,aq,aq^2)_\infty}\left(\sum_{r\geq 0}q^r\alpha_r + \sum_{\substack{r\geq 0\\ n\geq 1}}(-1)^n a^{n-1}q^{\binom{n+1}{2}+nr}(1+aq^r)\alpha_r\right). \tag{1.11}$$

(8) *If* (α_n, β_n) *is a Bailey pair relative to* (a^2, q), *then*

$$\sum_{n\geq 0} \frac{(a^2)_{2n} q^n}{(a, aq)_n} \beta_n = \frac{1}{(q, aq, aq)_\infty} \sum_{n,r\geq 0} \frac{(-a)^n q^{\binom{n+1}{2}+nr+r}(1+a)}{1+aq^r} \alpha_r. \tag{1.12}$$

(9) *If* (α_n, β_n) *is a Bailey pair relative to* (a^2, q), *then*

$$\sum_{n\geq 0} \frac{(a^2q)_{2n} q^n}{(aq, aq)_n} \beta_n = \frac{1}{(q, aq, aq)_\infty} \sum_{n_1,n_2,r\geq 0} (-a)^{n_1+n_2} q^{\binom{n_1+1}{2}+\binom{n_2+1}{2}+n_1r+n_2r+r} \alpha_r. \tag{1.13}$$

(10) *If* (α_n, β_n) *is a Bailey pair relative to* (a^2, q), *then*

$$\sum_{n\geq 0} (-aq)_n q^n \beta_n = \frac{(-aq)_\infty}{(q, a^2q)_\infty} \left(\sum_{r\geq 0} q^r \alpha_r - \sum_{\substack{n\geq 1\\ r\geq 0}} a^{3n-2} q^{n(3n-1)/2+3nr-r} (1+aq^r)(1-aq^{n+r}) \alpha_r \right). \tag{1.14}$$

(11) *If* (α_n, β_n) *is a Bailey pair relative to* (a, q), *then*

$$\sum_{n\geq 0} q^{2n} \beta_n = \frac{1}{(aq, q)_\infty} \left(\sum_{r\geq 0} q^{2r} \alpha_r + \sum_{\substack{n\geq 1\\ r\geq 0}} (-1)^n a^{n-1} q^{\binom{n+1}{2}+2nr} (1+aq^{2r}) \alpha_r \right). \tag{1.15}$$

(12) *If* (α_n, β_n) *is a Bailey pair relative to* (a, q), *then*

$$\sum_{n\geq 0} (aq; q^2)_n q^{2n} \beta_n = \frac{1}{(q)_\infty (aq^2; q^2)_\infty (1+q)} \sum_{r,n\geq 0} (-a)^n q^{n^2+n+2rn+2r} (1-q^{2n+2}) \alpha_r. \tag{1.16}$$

(13) *If* (α_n, β_n) *is a Bailey pair relative to* (a^2q, q), *then*

$$\sum_{n\geq 0} (-aq)_n q^n \beta_n = \frac{(-aq)_\infty}{(a^2q^2, q)_\infty} \sum_{n,r\geq 0} a^{3n} q^{n(3n+5)/2+r+3nr} (1-aq^{n+r+1}) \alpha_r. \tag{1.17}$$

(14) *If* (α_n, β_n) *is a Bailey pair relative to* (a, q), *then*

$$\sum_{n\geq 0} (aq^2; q^2)_n q^n \beta_n = \frac{1}{(aq; q^2)_\infty (q)_\infty (1+q)} \sum_{n,r\geq 0} (-a)^n q^{n^2+2rn+r} (1+q^{2n+1}) \alpha_r. \tag{1.18}$$

(15) *If (α_n,β_n) is a Bailey pair relative to (a^2q,q), then*

$$\begin{aligned}\sum_{n\geq 0}\frac{(a^2q^2)_{2n}q^n\beta_n}{(aq)_n} &= \frac{1}{(aq,q)_\infty}\sum_{n.r\geq 0} a^{3n-1}q^{3n^2+3nr+n-1}(1-q^n)\alpha_r \\ &+ \frac{1}{(aq,q)_\infty}\sum_{n.r\geq 0} a^{3n}q^{3n^2+3nr+4n+r}(1-aq^{n+r+1})\alpha_r.\end{aligned} \tag{1.19}$$

(16) *If (α_n,β_n) is a Bailey pair relative to (a^2q,q), then*

$$\sum_{n\geq 0}\frac{(a^2q^2)_{2n}q^n\beta_n}{(aq^2,aq)_n} = \frac{1}{(aq^2,aq,q)_\infty}\sum_{n_1,n_2,r\geq 0}(-a)^{n_1+n_2}q^{\binom{n_1}{2}+\binom{n_2+1}{2}+(r+1)n_1+(r+1)n_2+r}\alpha_r. \tag{1.20}$$

(17) *If (α_n,β_n) is a Bailey pair relative to (a^2,q^2), then*

$$\sum_{n\geq 0}(-aq)_{2n}q^{2n}\beta_n = \frac{1}{(aq)_\infty(q^2;q^2)_\infty(1-q)}\sum_{n,r\geq 0}(-a)^n q^{\binom{n+1}{2}+2n+2r}(1-q^{n+1})\alpha_r. \tag{1.21}$$

There are also some sporadic identities which lead to nice statements about Bailey pairs, and we shall briefly touch on two of these. We have the following.

Theorem 1.2. (1) *If (α_n,β_n) is a Bailey pair relative to (a,q) and if (α'_n,β'_n) is a Bailey pair relative to (z,q), then*

$$\sum_{n,s\geq 0}a^s z^n q^{2ns+s}\beta_n\beta'_s = \frac{1}{(aq,z)_\infty}\sum_{n,r\geq 0}\frac{a^n z^r q^{2nr+n}(1-z)}{1-zq^{2n}}\alpha_r\alpha'_n. \tag{1.22}$$

(2) *If (α_n,β_n) is a Bailey pair relative to (a,q) and (α'_n,β'_n) is a Bailey pair relative to (z,q), then*

$$\sum_{n,s\geq 0}(aq)_{2n}a^s z^{n+s}q^{s^2+s+2ns}\beta_n\beta'_s = \frac{1}{(z)_\infty}\sum_{n,r\geq 0}\frac{a^n z^{r+n}q^{n^2+n+2rn}(1-z)}{1-zq^{2n}}\alpha_r\alpha'_n. \tag{1.23}$$

Two corollaries in the spirit of Theorem 1.1 are:

Corollary 1.3. (1) *If (α_n,β_n) is a Bailey pair relative to (a,q), then*

$$\sum_{n\geq 0}(aq)_{2n}q^n\beta_n = \frac{1}{(q)_\infty}\sum_{r,n\geq 0}(-a)^n q^{3n(n+1)/2+(2n+1)r}\alpha_r. \tag{1.24}$$

(2) *If (α_n,β_n) is a Bailey pair relative to (a,q), then*

$$\sum_{n\geq 0}\frac{(aq)_{2n}q^n}{(-aq;q^2)_{n+1}}\beta_n = \frac{1}{(-aq;q^2)_\infty(q)_\infty}\sum_{n,r\geq 0}(-a)^n q^{2n^2+2n+2rn+r}\alpha_r. \tag{1.25}$$

Before proceeding, a number of remarks in order. First, for other work on conjugate Bailey pairs we refer to [10, 19, 20, 21]. The conjugate Bailey pair corresponding to (1.5) may be found in [10] and [21].

Second, our identities are inspired by recent work of Andrews and Warnaar [8], who used the partial theta identities

$$\sum_{n\geq 0}\frac{(a^2q^2;q^4)_n(a^2q^2;q^2)_nq^n}{(a^2q^2,-aq^3,q^2;q^2)_n} = \frac{(-q)_\infty}{(-aq)_\infty}\sum_{n\geq 0}a^nq^{n^2} \tag{1.26}$$

and

$$\sum_{n\geq 0} \frac{(aq)_{2n}q^n}{(a^2q^2,q^2;q^2)_n} = \frac{(-q)_\infty}{(-aq)_\infty} \sum_{n\geq 0} a^n q^{\binom{n+1}{2}} \tag{1.27}$$

to deduce that if (α_n, β_n) is a Bailey pair relative to (a^2, q^2), then

$$\sum_{n\geq 0} \frac{(a^2q^2;q^2)_{2n}q^n}{(-aq)_{2n+1}} \beta_n = \frac{(-q)_\infty}{(-aq)_\infty} \sum_{r,n\geq 0} a^n q^{n^2+n+2rn+r} \alpha_r \tag{1.28}$$

and

$$\sum_{n\geq 0} (aq)_{2n} q^n \beta_n = \frac{(-q)_\infty}{(-aq)_\infty} \sum_{r,n\geq 0} a^n q^{\binom{n+1}{2}+2rn+r} \alpha_r. \tag{1.29}$$

The question of whether there was a systematic way to produce "good" partial theta identities like (1.26) and (1.27) led us to the residual identities of Ramanujan-type partial theta identities.

Third, we note that expressions involving binary quadratic forms are especially important in the study of q-series. This is principally due to their relation (in the case of indefinite forms) to Ramanujan's mock theta functions [2, 7, 14, 25], though they arise elsewhere as well (e.g. [6, 15, 16]). Our results can be used to obtain many q-series which are *mixed mock modular forms*, i.e., functions of the form $\sum_{i=1}^k f_i g_i$, where f_i is a modular form and g_i is a mock modular form [11, 24]. We'll give just one example, but there are many more. Consider the Bailey pair relative to $(1, q)$ [4, Lemma 3.3]

$$\alpha_n = \begin{cases} (-1)^n \left(z^n q^{\binom{n}{2}} + z^{-n} q^{\binom{n+1}{2}} \right), & n > 0, \\ 1, & n = 0, \end{cases}$$

and

$$\beta_n = \frac{(z)_n (q/z)_n}{(q)_{2n}}.$$

Substituting into (1.24) and simplifying gives

$$\sum_{n\geq 0} q^n (z, q/z)_n = \frac{1}{(q)_\infty} \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) (-1)^{n+r} z^r q^{3n(n+1)/2+r(r+1)/2+2nr}.$$

We shall not go in detail here, but it follows from the work of Zwegers [25] or Hickerson and Mortenson [14] that these yield mixed mock modular forms.

Finally, while speaking of applications we should not overlook the infinite product expansions obtained by applying the unit Bailey pair relative to (a, q) [3, (3.47) and (3.48)],

$$\alpha_n = \frac{(a)_n (1 - aq^{2n}) (-1)^n q^{\binom{n}{2}}}{(q)_n (1 - a)}$$

and

$$\beta_n = \delta_{n,0},$$

to any of the instances of Theorem 1.1 or Corollary 1.3. When $a = 1$ or q the α_n simplify nicely and the product expansions are particularly elegant.

2. Proof of Theorem 1.1

In this section we prove the identities in Theorem 1.1. In each case, we start with a Ramanujan-type partial theta identity, compute the corresponding residual identity, and then deduce a statement about Bailey pairs. The argument is similar in each case so we'll only do one of them in detail.

Proof of Theorem 1.1. We begin by looking at partial theta identities recorded by Ramanujan in his lost notebook, as presented in [5]. Entry 6.3.2 reads

$$\begin{aligned}\sum_{n\geq 0}\frac{q^n}{(aq,q/a)_n} &= (1-a)\sum_{n\geq 0}(-1)^n a^{3n}q^{n(3n+1)/2}(1-a^2q^{2n+1})\\ &\quad + \frac{1}{(aq,q/a)_\infty}\sum_{n\geq 0}(-1)^n a^{2n+1}q^{n(n+1)/2}.\end{aligned} \tag{2.1}$$

The idea of the residual identity goes back to Andrews [1]. We fix a positive integer N, multiply both sides of (2.1) by $(1-q^N/a)$, and compute the limit as $a\to q^N$. The summand on the left is then 0 for $n<N$ so we may shift the index of summation by $n\to n+N$ and use $(x)_{n+N}=(x)_N(xq^N)_n$ to simplify. The result is

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{n+N}}{(q^{2N+1},q)_n} &= (q^{N+1},q^{1-N})_N(1-q^N)\sum_{n\geq 0}(-1)^n q^{3Nn}q^{n(3n+1)/2}(1-q^{2N}q^{2n+1})\\ &\quad + \frac{1}{(q^{2N+1},q)_\infty}\sum_{n\geq 0}(-1)^n q^{2Nn+N}q^{n(n+1)/2}\\ &= \frac{1}{(q^{2N+1},q)_\infty}\sum_{n\geq 0}(-1)^n q^{2Nn+N}q^{n(n+1)/2}.\end{aligned} \tag{2.2}$$

Since this identity is true for any $N>0$, analytic continuation allows us to replace q^N by a. The result (after replacing a^2 by a) is the residual identity

$$\sum_{n\geq 0}\frac{q^n}{(aq,q)_n} = \frac{1}{(aq,q)_\infty}\sum_{n\geq 0}(-1)^n a^n q^{n(n+1)/2}. \tag{2.3}$$

(This is also a consequence of the Heine transformation. See (3.2).)

To deduce part (1) of Theorem 1.1 is now an easy matter. Beginning with (1.2) we have

$$\begin{aligned}\sum_{n\geq 0}q^n\beta_n &= \sum_{n\geq 0}q^n\sum_{r=0}^{n}\frac{\alpha_r}{(aq)_{n+r}(q)_{n-r}}\\ &= \sum_{r\geq 0}\alpha_r\sum_{n=r}^{\infty}\frac{q^n}{(aq)_{n+r}(q)_{n-r}}\\ &= \sum_{r\geq 0}\alpha_r\sum_{n\geq 0}\frac{q^{n+r}}{(aq)_{n+2r}(q)_n}\\ &= \sum_{r\geq 0}\frac{q^r\alpha_r}{(aq)_{2r}}\sum_{n\geq 0}\frac{q^n}{(aq^{2r+1})_n(q)_n}\\ &= \sum_{r\geq 0}q^r\alpha_r\frac{1}{(aq^{2r+1})_\infty(q)_\infty}\sum_{n\geq 0}(-1)^n a^n q^{2rn+\binom{n+1}{2}}\\ &= \frac{1}{(aq,q)_\infty}\sum_{r,n\geq 0}(-a)^n q^{\binom{n+1}{2}+2rn+r}\alpha_r,\end{aligned}$$

the penultimate step being an application of (2.3) with $a=aq^{2r}$.

We continue with Entry 6.3.6, which reads

$$\begin{aligned}\left(1+\frac{1}{a}\right)\sum_{n\geq 0}\frac{(q;q^2)_n q^{2n+1}}{(-aq,-q/a;q^2)_{n+1}} &= \sum_{n\geq 0}(-1)^n a^n q^{\binom{n+1}{2}} \\ &\quad - \frac{(q;q^2)_\infty}{(-aq,-q/a;q^2)_\infty}\sum_{n\geq 0} a^{3n}q^{3n^2+n}(1-a^2q^{4n+2}).\end{aligned} \tag{2.4}$$

The residual identity is (after setting $q = q^{1/2}$ and $a = a/q$)

$$\sum_{n\geq 0}\frac{(a)_n q^n}{(a^2q,q)_n} = \frac{(aq)_\infty}{(a^2q,q)_\infty}\sum_{n\geq 0}(-1)^n a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}). \tag{2.5}$$

The resulting Bailey statement is part (2) of Theorem 1.1.

Entry 6.3.7 reads

$$\begin{aligned}\left(1+\frac{1}{a}\right)\sum_{n\geq 0}\frac{(-q)_{2n} q^{2n+1}}{(aq,q/a;q^2)_{n+1}} &= -\sum_{n\geq 0}(-a)^n q^{n^2+n} \\ &\quad + \frac{(-q)_\infty}{(aq,q/a;q^2)_\infty}\sum_{n\geq 0}(-a)^n q^{\binom{n+1}{2}}.\end{aligned} \tag{2.6}$$

The residual identity is

$$\sum_{n\geq 0}\frac{(aq)_{2n} q^{2n}}{(a^2q^4,q^2;q^2)_n} = \frac{(aq^2)_\infty}{(a^2q^4,q^2;q^2)_\infty}\sum_{n\geq 0} a^n q^{\binom{n+1}{2}+n} \tag{2.7}$$

The Bailey statement is part (3) of Theorem 1.1.

Entry 6.3.9 reads

$$\begin{aligned}\sum_{n\geq 0}\frac{(q;q^2)_n q^{2n}}{(-aq^2,-q^2/a;q^2)_n} &= (1+a)\sum_{n\geq 0}(-a)^n q^{\binom{n+1}{2}} \\ &\quad - \frac{a(q;q^2)_\infty}{(-aq^2,-q^2/a;q^2)_\infty}\sum_{n\geq 0} a^{3n} q^{3n^2+2n}(1-aq^{2n+1}).\end{aligned} \tag{2.8}$$

The residual identity is

$$\sum_{n\geq 0}\frac{(aq)_n q^n}{(a^2q^2,q)_n} = \frac{(aq)_\infty}{(a^2q^2,q)_\infty}\sum_{n\geq 0}(-a)^{3n} q^{n(3n+5)/2}(1+aq^{n+1}). \tag{2.9}$$

This identity was also stated by Warnaar [23]. The Bailey statement is part (4) of Theorem 1.1.

Entry 6.3.11 reads

$$\begin{aligned}\sum_{n\geq 0}\frac{(q;q^2)_n q^n}{(-aq,-q/a)_n} &= (1+a)\sum_{n\geq 0}(-a)^n q^{\binom{n+1}{2}} \\ &\quad - \frac{a(q;q^2)_\infty}{(-aq,-q/a)_\infty}\sum_{n\geq 0}(-1)^n a^{2n} q^{n^2+n}.\end{aligned} \tag{2.10}$$

The residual identity is

$$\sum_{n\geq 0}\frac{(aq;q^2)_n q^n}{(aq,q)_n} = \frac{(aq;q^2)_\infty}{(aq,q)_\infty}\sum_{n\geq 0}(-a)^n q^{n^2+n}. \tag{2.11}$$

The Bailey statement is part (5) of Theorem 1.1.

Entry 6.6.1 is (1.1). The residual identity is

$$\sum_{n\geq 0}\frac{(a^2q^{n+1})_nq^n}{(aq,q)_n}=\frac{1}{(aq,q)_\infty}\sum_{n\geq 0}a^{3n}q^{3n^2+2n}(1-aq^{2n+1}). \tag{2.12}$$

This was stated by both Andrews [1] and Warnaar [23]. The Bailey statement is part (6) of Theorem 1.1.

We now leave Ramanujan and turn to some partial theta identities established by Warnaar [23]. First, his equation (3.9) is

$$\sum_{n\geq 0}\frac{(q^{n+1})_nq^n}{(1-a)(aq,q^2,q/a)_n}=1+\frac{a+(1+a)\sum_{n\geq 1}(-a)^nq^{\binom{n+1}{2}}}{(q,a,a/q)_\infty}, \tag{2.13}$$

and on p. 389 he computes the residual identity (p.389),

$$\sum_{n\geq 0}\frac{(a^2q^{n+1})_nq^n}{(q,aq,aq^2)_n}=\frac{1+(1+1/a)\sum_{n\geq 1}(-a)^nq^{\binom{n+1}{2}}}{(1-q)(q,aq,aq^2)_\infty}. \tag{2.14}$$

The Bailey statement is part (7) of Theorem 1.1.

Next, in his equation (4.10) Warnaar states the partial theta identity

$$1+2\sum_{n\geq 1}\frac{(q^n)_nq^n}{(q,aq,q/a)_n}=\frac{1-a}{1+a}\left(1-2\sum_{n\geq 1}\frac{(-a)^nq^{\binom{n}{2}}}{(q,a,q/a)_\infty}\right), \tag{2.15}$$

and on p. 389 he calculates the corresponding residual identity,

$$\sum_{n\geq 0}\frac{(a^2)_{2n}q^n}{(q,a,aq,a^2q)_n}=\frac{1}{(q,aq,aq)_\infty}\sum_{n\geq 0}(-a)^nq^{\binom{n+1}{2}}. \tag{2.16}$$

The resulting Bailey statement is part (8) of Theorem 1.1.

In his equation (4.13), Warnaar states the partial theta identity

$$\sum_{n\geq 0}\frac{(q^{n+1})_nq^n}{(a)_{n+1}(q,q/a)_n}=\sum_{n\geq 0}(-a)^nq^{\binom{n+1}{2}}+a\frac{\left(\sum_{n\geq 0}(-a)^nq^{\binom{n+1}{2}}\right)^2}{(q,a,q/a)_\infty}, \tag{2.17}$$

and on p. 389 he computes the corresponding residual identity,

$$\sum_{n\geq 0}\frac{(a^2q^{n+1})_nq^n}{(q,aq,aq)_n}=\frac{1}{(q,aq,aq)_\infty}\left(\sum_{n\geq 0}(-a)^nq^{\binom{n+1}{2}}\right)^2. \tag{2.18}$$

The resulting Bailey statement is part (9) of Theorem 1.1.

In his equation (4.15), Warnaar states the partial theta identity

$$\sum_{n\geq 0}\frac{(-q)_nq^n}{(a)_{n+1}(q/a)_n}=1+(1+a)\sum_{n\geq 1}(-1)^na^{2n-1}q^{n^2}+\frac{a-(1+a)\sum_{n\geq 1}a^{3n-1}q^{n(3n-1)/2}(1-aq^n)}{(q;q^2)_\infty(a,q/a)_\infty}, \tag{2.19}$$

and on p. 390 he deduces the residual identity

$$\sum_{n\geq 0}\frac{(-aq)_nq^n}{(q,a^2q)_n}=\frac{(-aq)_\infty}{(q,a^2q)_\infty}\left(1-(1+a)\sum_{n\geq 1}a^{3n-2}q^{n(3n-1)/2}(1-aq^n)\right). \tag{2.20}$$

The Bailey statement is part (10) of Theorem 1.1.

Finally, on the top of p. 379 Warnaar proves

$$\sum_{n\geq 0}\frac{q^{2n}}{(a)_{n+1}(q/a)_n}=1+a+(1+a^2)\sum_{n\geq 1}(-1)^n a^{3n-2}q^{n(3n-1)/2}(1+aq^n)+\frac{a^2+(1+a^2)\sum_{n\geq 1}(-1)^n a^{2n}q^{\binom{n+1}{2}}}{(a,q/a)_\infty}. \tag{2.21}$$

The corresponding residual identity is

$$\sum_{n\geq 0}\frac{q^{2n}}{(a^2q,q)_n}=\frac{1}{(a^2q,q)_\infty}+\frac{1+1/a^2}{(a^2q,q)_\infty}\sum_{n\geq 1}(-1)^n a^{2n}q^{\binom{n+1}{2}}. \tag{2.22}$$

The resulting Bailey statement is part (11) of Theorem 1.1.

For the remaining six parts of Theorem 1.1 we use Warnaar's framework to compute more Ramanujan-type partial theta identities. Namely, we use a Bailey pair relative to (q,q) or (q^2,q) in either Corollary 4.1 or Corollary 7.1 of [23]. To obtain Bailey pairs we will employ Slater's list [22], the fact [18, $b=0$] that if (α_n,β_n) is a Bailey pair relative to (a,q), then (α_n^*,β_n^*) is a Bailey pair relative to (aq,q), where

$$\alpha_n^*=\frac{(1-aq^{2n+1})a^nq^{n^2+n}}{(1-aq)}\sum_{r=0}^{n}a^{-r}q^{-r^2}\alpha_r \tag{2.23}$$

and

$$\beta_n^*=\beta_n, \tag{2.24}$$

and the fact [3, Theorem 3.3, (3.47) and (3.48)] that (α_n,β_n) is a Bailey pair relative to (a,q), where

$$\alpha_n=\frac{(1-aq^{2n})(a,b,c)_n(-1)^nq^{\binom{n}{2}}(\frac{aq}{bc})^n}{(1-a)(q,aq/b,aq/c)_n} \tag{2.25}$$

and

$$\beta_n=\frac{(aq/bc)_n}{(q,aq/b,aq/c)_n}. \tag{2.26}$$

We begin with the corrected pair $E(6)$ from Slater's list of Bailey pairs [22]. We have that (α_n,β_n) is a Bailey pair relative to (q,q), where

$$\alpha_n=\frac{(-1)^nq^{n^2-n}(1-q^{4n+2})}{1-q^2}$$

and

$$\beta_n=\frac{q^n}{(q^2;q^2)_n}.$$

Inserting this into Corollary 4.1 of [23] and simplifying, the resulting partial theta identity is

$$\sum_{n\geq 0}\frac{(q;q^2)_nq^{2n}}{(aq,q/a)_n}=\frac{(1-a)}{(1+q)}\sum_{n\geq 0}a^nq^{\binom{n}{2}}(1+q^{2n+1})+\frac{(q;q^2)_\infty}{(aq,q/a)_\infty(1+q)}\sum_{n\geq 1}(-1)^{n-1}a^{2n}q^{n^2-n}(1-q^{2n}). \tag{2.27}$$

The corresponding residual identity is

$$\sum_{n\geq 0}\frac{(aq;q^2)_nq^{2n}}{(aq,q)_n}=\frac{1}{(q)_\infty(aq^2;q^2)_\infty(1+q)}\sum_{n\geq 0}(-a)^nq^{n^2+n}(1-q^{2n+2}). \tag{2.28}$$

The Bailey statement is part (12) of Theorem 1.1.

Next from Slater's $C(3)$ and (2.23), (2.24) (we omit the details) we find the Bailey pair relative to (q^2,q),

$$\alpha_n=\frac{(1-q^{2n+2})}{(1-q^2)}q^{n^2+n-(n/2)(n/2+1)}(-1)^{n/2}\chi(n\text{ even})$$

and

$$\beta_n = \frac{1}{(q^3;q^2)_\infty (q)_\infty}.$$

Using Corollary 7.1 of [23] the resulting partial theta identity is

$$\sum_{n\geq 0} \frac{(-q)_n q^n}{(aq)_n (q/a)_{n+1}} + (1-a)\sum_{n\geq 0}(-1)^n q^{n^2-1} a^{2n+1} = \frac{(-q)_\infty}{(aq,q/a)_\infty} \sum_{n\geq 0} a^{3n+1} q^{n(3n-1)/2-1}(1-aq^n). \quad (2.29)$$

The corresponding residual identity is

$$\sum_{n\geq 0} \frac{(-a)_n q^n}{(a^2,q)_n} = \frac{(-a)_\infty}{(a^2,q)_\infty} \sum_{n\geq 0} a^{3n} q^{n(3n-1)/2}(1-aq^n). \quad (2.30)$$

The Bailey statement is part (13) of Theorem 1.1.

From (2.25) and (2.26) with $b=-q^2$ and $c\to\infty$ we obtain the Bailey pair relative to (q^2,q),

$$\alpha_n = (-1)^n q^{n^2} \frac{(1-q^{2n+2})^2}{(1-q^2)^2}$$

and

$$\beta_n = \frac{1}{(q^2;q^2)_n}.$$

Using Corollary 7.1 of [23] the resulting partial theta identity is

$$\begin{aligned} &\sum_{n\geq 0} \frac{(q^3;q^2)_n q^n}{(aq)_n (q/a)_{n+1}} + \frac{(1-a)}{(1-q^2)} \sum_{n\geq 0} a^{n+1} q^{\binom{n}{2}-1}(1-q^{2n+2}) \\ &\qquad = \frac{(q;q^2)_\infty}{(aq,q/a)_\infty (1-q^2)} \sum_{n\geq 0} a^{2n+1} q^{n^2-n-1}(-1)^n (1+q^{2n+1}). \end{aligned} \quad (2.31)$$

The corresponding residual identity is

$$\sum_{n\geq 0} \frac{(aq^2;q^2)_n q^n}{(aq,q)_n} = \frac{1}{(aq;q^2)_\infty (q)_\infty (1+q)} \sum_{n\geq 0} (-a)^n q^{n^2}(1+q^{2n+1}). \quad (2.32)$$

The Bailey statement is part (14) of Theorem 1.1.

From (2.25) and (2.26) with $b,c\to\infty$ we obtain the Bailey pair relative to (q^2,q),

$$\alpha_n = (-1)^n q^{3n(n+1)/2} \frac{(1-q^{2n+2})(1-q^{n+1})}{(1-q^2)(1-q)}$$

and

$$\beta_n = \frac{1}{(q)_n}.$$

Using Corollary 7.1 of [23] the resulting partial theta identity is

$$\begin{aligned} &\sum_{n\geq 0} \frac{(q^{n+1})_{n+1} q^n}{(aq)_n (q/a)_{n+1}} + (1-a)\sum_{n\geq 0} a^{n+1} q^{n^2+n-1}(1-q^{n+1}) \\ &\qquad = \frac{1}{(aq,q/a)_\infty} \left(\sum_{n\geq 0} a^{3n} q^{3n^2-2n-1}(1-q^n) + \sum_{n\geq 0} a^{3n+1} q^{3n^2+r-1}(1-aq^n) \right). \end{aligned} \quad (2.33)$$

The residual identity is

$$\sum_{n\geq 0} \frac{(a^2q^n)_n q^n}{(a,q)_n} = \frac{1}{(a,q)_\infty} \left(\sum_{n\geq 0} a^{3n-1} q^{3n^2-2n}(1-q^n) + \sum_{n\geq 0} a^{3n} q^{3n^2+n}(1-aq^n) \right). \quad (2.34)$$

The Bailey statement is part (15) of Theorem 1.1.

From (2.25) and (2.26) we obtain the Bailey pair relative to (q^2, q),

$$\alpha_n = q^{n^2+n}\frac{(1-q^{2n+2})}{(1-q^2)}$$

and

$$\beta_n = \frac{1}{(q^2, q)_n}$$

Using Corollary 7.1 of [23] the resulting partial theta identity is

$$\sum_{n\geq 0}\frac{(q^{n+2})_n q^{n+1}}{(aq,q)_n(q/a)_{n+1}} + (1-a)\sum_{n\geq 0}(-1)^n a^{n+1} q^{\binom{n+1}{2}} = \frac{a}{(q,aq,q/a)_\infty}\sum_{n_1,n_2\geq 0}(-a)^{n_1+n_2} q^{\binom{n_1}{2}+\binom{n_2+1}{2}}. \tag{2.35}$$

The residual identity is

$$\sum_{n\geq 0}\frac{(a^2q^n)_n q^n}{(aq,a,q)_n} = \frac{1}{(aq,a,q)_\infty}\sum_{n_1,n_2\geq 0}(-a)^{n_1+n_2} q^{\binom{n_1}{2}+\binom{n_2+1}{2}}. \tag{2.36}$$

The corresponding Bailey statement is part (16) of Theorem 1.1.

Finally, from (2.25) and (2.26) we obtain the Bailey pair relative to (q^4, q^2),

$$\alpha_n = q^{2n^2+n}\frac{(1-q^{4n+4})(1-q^{2n+2})}{(1-q^4)(1-q^2)}$$

and

$$\beta_n = \frac{1}{(q^2)_{2n}}.$$

Using Corollary 7.1 of [23] one more time, the partial theta identity is

$$\begin{aligned}\sum_{n\geq 0}\frac{(-q)_{2n+1}q^{2n+2}}{(aq^2;q^2)_n(q^2/a;q^2)_{n+1}} &+ \frac{(1-a)}{(1-q)}\sum_{n\geq 0}(-1)^n a^{n+1} q^{n^2}(1-q^{2n+2}) \\ &= \frac{(-q)_\infty}{(aq^2,q^2/q;q^2)_\infty(1-q)}\sum_{n\geq 0}(-1)^n a^{n+1} q^{\binom{n}{2}}(1-q^{n+1}).\end{aligned} \tag{2.37}$$

The residual identity is

$$\sum_{n\geq 0}\frac{(-a)_{2n}q^{2n}}{(a^2,q^2;q^2)_n} = \frac{1}{(a)_\infty(q^2;q^2)_\infty(1-q)}\sum_{n\geq 0}(-a)^n q^{\binom{n}{2}}(1-q^{n+1}). \tag{2.38}$$

The corresponding Bailey statement is part (17) of Theorem 1.1. This completes the proof. □

3. Proof of Theorem 1.2 and Corollary 1.3

To prove the two identities in Theorem 1.2, we shall use the standard argument twice, once with respect to (1.2) and once with respect to the Bailey pair inversion [3, (3.40)],

$$\alpha_n = (1-aq^{2n})\sum_{j=0}^{n}\frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}\beta_j}{(q)_{n-j}}. \tag{3.1}$$

Proof of Theorem 1.2. We begin with a special case of the Heine transformation [13, p. 359, Eq. (III.1), $a, b \to 0$, $c = aq$],

$$\sum_{n\geq 0}\frac{z^n}{(aq,q)_n} = \frac{1}{(aq,z)_\infty}\sum_{n\geq 0}\frac{(z)_n(-a)^n q^{\binom{n+1}{2}}}{(q)_n}. \tag{3.2}$$

Arguing as usual, the corresponding Bailey statement is that if (α_n, β_n) is a Bailey pair relative to (a, q), then

$$\sum_{n\geq 0} z^n \beta_n = \frac{1}{(aq,z)_\infty} \sum_{r,n\geq 0} \frac{(z)_n(-a)^n z^r q^{\binom{n+1}{2}+2rn}}{(q)_n} \alpha_r. \tag{3.3}$$

(Notice that this is a generalization of part (1) of Theorem 1.1. It is originally due to Bressoud [10] and Singh [21].) Now using (3.3) along with the Bailey pair inversion (3.1) we may deduce part (1) of Theorem 1.2 as follows:

$$\begin{aligned}
\frac{1}{(aq,z)_\infty} \sum_{r,n\geq 0} \frac{a^n z^r q^{2rn+n}(1-z)}{(1-zq^{2n})} \alpha_r \alpha'_n &= \frac{1}{(aq,z)_\infty} \sum_{r,n\geq 0} \frac{a^n z^r q^{2rn+n}(1-z)}{(1-zq^{2n})} \alpha_r \\
&\quad \times \frac{(1-zq^{2n})}{(1-z)} \sum_{s=0}^{n} \frac{(z)_{n+s}(-1)^{n-s} q^{\binom{n-s}{2}} \beta'_s}{(q)_{n-s}} \\
&= \frac{1}{(aq,z)_\infty} \sum_{r,s\geq 0} z^r \alpha_r \beta'_s \sum_{n=s}^{\infty} \frac{(z)_{n+s}(-1)^{n-s} a^n q^{\binom{n-s}{2}+n+2rn}}{(q)_{n-s}} \\
&= \frac{1}{(aq,z)_\infty} \sum_{r,s\geq 0} z^r \alpha_r a^s q^{2rs+s} (z)_{2s} \beta'_s \sum_{n=0}^{\infty} \frac{(z^{2s})_n (-a)^n q^{\binom{n+1}{2}+2rn}}{(q)_n} \\
&= \sum_{s\geq 0} a^s q^s \beta'_s \frac{1}{(aq, zq^{2s})_\infty} \sum_{r,n\geq 0} \frac{z^r q^{2rs} (zq^{2s})_n (-a)^n q^{\binom{n+1}{2}+2rn} \alpha_r}{(q)_n} \\
&= \sum_{s\geq 0} a^s q^s \beta'_s \sum_{n\geq 0} z^n q^{2ns} \beta_n,
\end{aligned}$$

the last step being an application of (3.3) with $z = zq^{2s}$.

Next we consider a special case of an identity of Fine [12, Eq. (25.96), $b = 0, t = z$],

$$\sum_{n\geq 0} \frac{(aq)_{2n} z^n}{(aq,q)_n} = \frac{1}{(z)_\infty} \sum_{n\geq 0} \frac{(z)_n(-az)^n q^{n(3n+1)/2}}{(q)_n}. \tag{3.4}$$

The corresponding Bailey statement is that if (α_n, β_n) is a Bailey pair relative to (a, q), then

$$\sum_{n\geq 0} (aq)_{2n} z^n \beta_n = \frac{1}{(z)_\infty} \sum_{r,n\geq 0} \frac{(z)_n(-a)^n z^{n+r} q^{n(3n+1)/2+2nr}}{(q)_n} \alpha_r \tag{3.5}$$

Arguing as in the case of part (1), from (3.5) and the Bailey pair inversion (3.1) we obtain part (2). □

We remark that (1.23) may also be deduced without much trouble from (1.22) by using the fact that if (α_n, β_n) is a Bailey pair relative to (z, q), then so is (α'_n, β'_n), where

$$\alpha'_n = z^n q^{n^2} \alpha_n$$

and

$$\beta'_n = \sum_{r=0}^{n} \frac{z^r q^{r^2}}{(q)_{n-r}} \beta_r.$$

This observation is due to O. Warnaar.

Proof of Corollary 1.3. For part (1) we simply set $z = q$ in (3.5). Next in part (2) of Theorem 1.2 we put the Bailey pair relative to q,

$$\alpha'_n = \frac{(-1)^n q^{n^2}(1-q^{2n+1})}{1-q}$$

and

$$\beta'_n = \frac{1}{(q^2;q^2)_n}.$$

The sum over s on the left-hand side of (1.23) then becomes a product by the q-binomial theorem [13, p. 354, (II.3)] and we obtain part (2) of Corollary 1.3. □

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