Generalizations of an identity of de Montmort

Michael J. Schlosser
Faculty of Mathematics
universität wien

joint work with

Tom H. Koornwinder
Korteweg-de Vries Institute for Mathematics
Universiteit van Amsterdam
See


Outline

1. De Montmort’s identity
2. Chaundy & Bullard’s proof
3. Daubechies’ proof
4. A proof by generating functions
5. Weighted lattice path enumeration
6. A proof using the beta integral
7. $q$-Extensions
De Montmort’s identity

In the 1713 second edition of his book *Essay d'analyse sur les jeux de hazard* Pierre Rémond de Montmort implicitly gave the identity

\[
1 = (1 - x)^n + \sum_{k=0}^{n} \binom{n+k}{k} x^k + x^{m+1} \sum_{k=0}^{m} \binom{m+k}{k} (1 - x)^k,
\]

where \( m, n \in \mathbb{N}_0 \).

Hereby he gave a new solution of the problem of points for two players (of unequal chances).

A solution of this problem for two players of equal chances (corresponding to \( x = \frac{1}{2} \)) was already given by Fermat and Pascal in 1654. In the case of two players of unequal chances a first solution (different from the one above) was given by Johann Bernoulli in 1710, in a letter to de Montmort.

Generalizations of an identity of de Montmort
De Montmort’s identity

In the 1713 second edition of his book *Essay d’analyse sur les jeux de hazard*, Pierre Rémond de Montmort implicitly gave the identity

\[
1 = (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n + k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m + k}{k} (1 - x)^k, \quad (dM)
\]

where \( m, n \in \mathbb{N}_0 \).
In the 1713 second edition of his book *Essay d’analyse sur les jeux de hazard*, Pierre Rémond de Montmort implicitly gave the identity

\[
1 = (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1-x)^k, \quad (dM)
\]

where \( m, n \in \mathbb{N}_0 \). Hereby he gave a new solution of the problem of points for two players (of unequal chances).
De Montmort’s identity

In the 1713 second edition of his book *Essay d’analyse sur les jeux de hazard*, Pierre Rémond de Montmort implicitly gave the identity

\[ 1 = (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n + k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m + k}{k} (1 - x)^k, \quad (dM) \]

where \( m, n \in \mathbb{N}_0 \). Hereby he gave a new solution of the problem of points for two players (of unequal chances).

A solution of this problem for two players of equal chances (corresponding to \( x = \frac{1}{2} \)) was already given by Fermat and Pascal in 1654.
De Montmort’s identity

In the 1713 second edition of his book *Essay d’analyse sur les jeux de hazard*, Pierre Rémond de Montmort implicitly gave the identity

\[ 1 = (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1 - x)^k, \quad (dM) \]

where \( m, n \in \mathbb{N}_0 \). Hereby he gave a new solution of the problem of points for two players (of unequal chances).

A solution of this problem for two players of equal chances (corresponding to \( x = \frac{1}{2} \)) was already given by Fermat and Pascal in 1654.

In the case of two players of unequal chances a first solution (different from the one above) was given by Johann Bernoulli in 1710, in a letter to de Montmort.
A probabilistic proof of de Montmort’s identity

Two players, Pierre and Paul play a game of chance. They shall have chances $x$ and $1-x$, respectively, of winning each round. Suppose the game is interrupted as soon as Pierre has won $m+1$ rounds or Paul has won $n+1$ rounds. Pierre is the winner, if he wins the last round, while before that he has already won $m$ rounds and Paul has won no more than $n$ rounds.

The chance for Pierre to win is $x^n \sum_{k=0}^{m+1} \binom{m+k}{k} x^m (1-x)^k$. Similarly, the chance for Paul to win is $(1-x)^m \sum_{k=0}^{n+1} \binom{n+k}{k} (1-x)^n x^k$.

These two chances necessarily add up to 1, thus (dM) follows.

Generalizations of an identity of de Montmort
A probabilistic proof of de Montmort's identity

Two players, Pierre and Paul play a game of chance. They shall have chances $x$ and $1-x$, respectively, of winning each round.
A probabilistic proof of de Montmort’s identity

Two players, Pierre and Paul play a game of chance.
They shall have chances $x$ and $1 - x$, respectively, of winning each round.
Suppose the game is interrupted as soon as Pierre has won $m + 1$ rounds or Paul has won $n + 1$ rounds.

Pierre is the winner, if he wins the last round, while before that he has already won $m$ rounds and Paul has won no more than $n$ rounds.

The chance for Pierre to win is

$$x^n \sum_{k=0}^{m} \binom{m + k}{k} x^m (1 - x)^k.$$  

Similarly, the chance for Paul to win is

$$(1 - x)^m \sum_{k=0}^{n} \binom{n + k}{k} (1 - x)^n x^k.$$  

These two chances necessarily add up to 1, thus (dM) follows.
Two players, Pierre and Paul play a game of chance. They shall have chances $x$ and $1-x$, respectively, of winning each round. Suppose the game is interrupted as soon as Pierre has won $m+1$ rounds or Paul has won $n+1$ rounds.

Pierre is the winner, if he wins the last round, while before that he has already won $m$ rounds and Paul has won no more than $n$ rounds.
A probabilistic proof of de Montmort’s identity

Two players, Pierre and Paul play a game of chance. They shall have chances \( x \) and \( 1 - x \), respectively, of winning each round. Suppose the game is interrupted as soon as Pierre has won \( m + 1 \) rounds or Paul has won \( n + 1 \) rounds. Pierre is the winner, if he wins the last round, while before that he has already won \( m \) rounds and Paul has won no more than \( n \) rounds.

The chance for Pierre to win is

\[
x \sum_{k=0}^{n} \binom{m+k}{k} x^m (1-x)^k.
\]
Two players, Pierre and Paul play a game of chance. They shall have chances $x$ and $1 - x$, respectively, of winning each round. Suppose the game is interrupted as soon as Pierre has won $m + 1$ rounds or Paul has won $n + 1$ rounds.

Pierre is the winner, if he wins the last round, while before that he has already won $m$ rounds and Paul has won no more than $n$ rounds.

The chance for Pierre to win is

$$x \sum_{k=0}^{n} \binom{m+k}{k} x^m (1-x)^k.$$ 

Similarly, the chance for Paul to win is

$$(1-x) \sum_{k=0}^{m} \binom{n+k}{k} (1-x)^n x^k.$$
A probabilistic proof of de Montmort’s identity

Two players, Pierre and Paul play a game of chance. They shall have chances $x$ and $1 - x$, respectively, of winning each round. Suppose the game is interrupted as soon as Pierre has won $m + 1$ rounds or Paul has won $n + 1$ rounds.

Pierre is the winner, if he wins the last round, while before that he has already won $m$ rounds and Paul has won no more than $n$ rounds.

The chance for Pierre to win is

$$x \sum_{k=0}^{n} \binom{m + k}{k} x^{m} (1 - x)^{k}.$$ 

Similarly, the chance for Paul to win is

$$(1 - x) \sum_{k=0}^{m} \binom{n + k}{k} (1 - x)^{n} x^{k}.$$ 

These two chances necessarily add up to 1, thus (dM) follows.
De Montmort’s identity can be written more succinctly as

\[ p_{m,n}(x) + p_{n,m}(1 - x) = 1, \]
De Montmort’s identity can be written more succinctly as

\[ p_{m,n}(x) + p_{n,m}(1-x) = 1, \]

where

\[ p_{m,n}(x) := (1-x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k = (1-x)^{n+1} \sum_{k=0}^{m} \frac{(n+1)_k}{k!} x^k \]

and

\[ (a)_k := \begin{cases} a(a+1) \ldots (a+k-1) & \text{if } k = 1, 2, \ldots, \\ 1 & \text{if } k = 0, \end{cases} \]

is the Pochhammer symbol.
This formula was rediscovered (partially or completely) several times:
This formula was rediscovered (partially or completely) several times:

- In 1738 De Moivre gives this identity in the second edition of his book *The doctrine of chances*, in his study of the *figurative numbers*. 
This formula was rediscovered (partially or completely) several times:

- In 1738 De Moivre gives this identity in the second edition of his book *The doctrine of chances*, in his study of the figurative numbers.

- In 1868 Hering (a high school teacher) derived the identity by manipulation of series and application of Pfaff’s $2F_1$ transformation.
This formula was rediscovered (partially or completely) several times:

- In 1738 De Moivre gives this identity in the second edition of his book *The doctrine of chances*, in his study of the figurative numbers.

- In 1868 Hering (a high school teacher) derived the identity by manipulation of series and application of Pfaff’s $_2F_1$ transformation.

- In 1960 Chaundy and Bullard rediscovered the identity in their solution of John Smith’s problem.
This formula was rediscovered (partially or completely) several times:

- In 1738 **De Moivre** gives this identity in the second edition of his book *The doctrine of chances*, in his study of the **figurative numbers**.

- In 1868 **Hering** (a high school teacher) derived the identity by **manipulation of series** and application of Pfaff’s $\,_{2}F_{1}$ transformation.

- In 1960 **Chaundy and Bullard** rediscovered the identity in their solution of **John Smith’s problem**.

- In 1971 **Herrmann** interpreted $p_{m,n}(x)$ as the polynomial of degree $m + n + 1$ which has a zero of order $n + 1$ at $x = 1$ and such that $1 - p_{m,n}(x)$ has a zero of order $m + 1$ at $x = 0$. He proved this by induction with respect to $m + n$. 
This formula was rediscovered (partially or completely) several times:

- In 1738 De Moivre gives this identity in the second edition of his book *The doctrine of chances*, in his study of the *figurative numbers*.

- In 1868 Hering (a high school teacher) derived the identity by manipulation of series and application of Pfaff’s $2F_1$ transformation.

- In 1960 Chaundy and Bullard rediscovered the identity in their solution of *John Smith’s problem*.

- In 1971 Herrmann interpreted $p_{m,n}(x)$ as the polynomial of degree $m + n + 1$ which has a zero of order $n + 1$ at $x = 1$ and such that $1 - p_{m,n}(x)$ has a zero of order $m + 1$ at $x = 0$. He proved this by induction with respect to $m + n$.

- In 1975 the (dM) identity was proposed for the *Canadian Mathematical Olympiad* (but not used there). Later, it appeared in the problem sections of *Crux Mathematicorum*, the *American Mathematical Monthly*, and *SIAM Review*. Different proofs by various people were given (induction; probabilistic; partial fractions).
In 1979 Jager observed that multiplication of both sides of (dM) by $x^{-m-1}(1-x)^{-n-1}$ yields the partial fraction decomposition

$$\frac{1}{x^{m+1}(1-x)^{n+1}} = \frac{r_{m,n}(x)}{x^{m+1}} + \frac{s_{m,n}(x)}{(1-x)^{n+1}},$$

with explicit $r_{m,n}(x)$ and $s_{m,n}(x)$ of respective degrees $m$ and $n$. 
In 1979 Jager observed that multiplication of both sides of \((dM)\) by \(x^{-m-1}(1-x)^{-n-1}\) yields the partial fraction decomposition

\[
\frac{1}{x^{m+1}(1-x)^{n+1}} = \frac{r_{m,n}(x)}{x^{m+1}} + \frac{s_{m,n}(x)}{(1-x)^{n+1}},
\]

with explicit \(r_{m,n}(x)\) and \(s_{m,n}(x)\) of respective degrees \(m\) and \(n\).

In 1986 Damjanovic, Klamkin & Ruehr observed an \(n\)-variable generalization of the identity:

\[
\sum_{i=1}^{n} x_i \sum_{k_1=0}^{a_1} \cdots \sum_{k_n=0}^{a_n} \delta_{k_i, a_i} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} x_1^{k_1} \cdots x_n^{k_n} = 1,
\]

where \(\sum_{i=1}^{n} x_i = 1\).
They gave a proof by generating functions and indicated a probabilistic proof.
In 1988 Daubechies rediscovered the $m = n$ case of (dM). This identity was a crucial step for her in order to arrive at the form of the function $m_0(\xi)$ which is associated with the wavelets of compact support named after her. Her proof utilizes Bézout’s identity.
In 1988 Daubechies rediscovered the $m = n$ case of (dM). This identity was a crucial step for her in order to arrive at the form of the function $m_0(\xi)$ which is associated with the wavelets of compact support named after her. Her proof utilizes Bézout’s identity.

Multiplication of both sides of (dM) by $(1 - x)^{-n-1}$ gives

$$(1 - x)^{-n-1} = \sum_{k=0}^{m} \frac{(n + 1)k}{k!} x^k + x^{m+1} \sum_{k=0}^{n} \frac{(m + 1)k}{k!} (1 - x)^{k-n-1}.$$ 

This identity is a special case of a three-term identity for three Gauß hypergeometric functions satisfying the same hypergeometric differential equation in a very degenerate case.
Chaundy & Bullard’s proof

Fix $m$ and $n$. By the binomial theorem we have

$$(x + y)^{m+n+1} = P_{m,n}(x,y) + x^{m+1}P_{n,m}(y,x),$$

where $P_{m,n}(x,y) := \sum_{k=0}^{m+n+1} \binom{m+n+1}{k} x^k y^{m+n+1-k}$ is a homogeneous polynomial of degree $m$.

Put $y = 1 - x$. Then

$$1 = (1 - x)^{n+1}P_{m,n}(x,1-x) + x^{m+1}P_{n,m}(1-x,x).$$

This identity was already obtained by Johann Bernoulli in 1710 in his solution of the problem of points.

Generalizations of an identity of de Montmort
Fix $m$ and $n$. By the binomial theorem we have
\[(x + y)^{m+n+1} = y^{n+1} P_{m,n}(x, y) + x^{m+1} P_{n,m}(y, x),\]
where
\[P_{m,n}(x, y) := \sum_{k=0}^{m} \binom{m+n+1}{k} x^k y^{m-k}\]
is a homogeneous polynomial of degree $m$. 
Fix $m$ and $n$. By the binomial theorem we have

$$(x + y)^{m+n+1} = y^{n+1}P_{m,n}(x, y) + x^{m+1}P_{n,m}(y, x),$$

where

$$P_{m,n}(x, y) := \sum_{k=0}^{m} \binom{m + n + 1}{k} x^k y^{m-k}$$

is a homogeneous polynomial of degree $m$.

Put $y = 1 - x$. Then

$$1 = (1 - x)^{n+1}P_{m,n}(x, 1-x) + x^{m+1}P_{n,m}(1-x, x).$$
Fix $m$ and $n$. By the binomial theorem we have

$$(x + y)^{m+n+1} = y^{n+1} P_{m,n}(x, y) + x^{m+1} P_{n,m}(y, x),$$

where

$$P_{m,n}(x, y) := \sum_{k=0}^{m} \binom{m+n+1}{k} x^k y^{m-k}$$

is a homogeneous polynomial of degree $m$.

Put $y = 1 - x$. Then

$$1 = (1 - x)^{n+1} P_{m,n}(x, 1 - x) + x^{m+1} P_{n,m}(1 - x, x).$$

This identity was already obtained by Johann Bernoulli in 1710 in his solution of the problem of points.
Multiplication of the last identity by \((1 - x)^{-n-1}\) yields

\[
(1 - x)^{-n-1} = P_{m,n}(x, 1 - x) + x^{m+1}(1 - x)^{-n-1}P_{n,m}(1 - x, x).
\]
Multiplication of the last identity by \((1 - x)^{-n-1}\) yields

\[
(1 - x)^{-n-1} = P_{m,n}(x, 1 - x) + x^{m+1}(1 - x)^{-n-1}P_{n,m}(1 - x, x).
\]

Expand both sides as a power series in \(x\). Then \(P_{m,n}(x, 1 - x)\) is a polynomial of degree \(\leq m\) in \(x\) and all terms in the power series of \(x^{m+1}(1 - x)^{-n-1}P_{n,m}(1 - x, x)\) have degree \(\geq m + 1\). Hence \(P_{m,n}(x, 1 - x)\) equals the power series of \((1 - x)^{-n-1}\) truncated after the term of \(x^m\), i.e.,

\[
P_{m,n}(x, 1 - x) = \sum_{k=0}^{m} \frac{(n + 1)k}{k!} x^k.
\]
Multiplication of the last identity by \((1 - x)^{-n-1}\) yields

\[
(1 - x)^{-n-1} = P_{m,n}(x, 1 - x) + x^{m+1}(1 - x)^{-n-1}P_{n,m}(1 - x, x).
\]

Expand both sides as a power series in \(x\). Then \(P_{m,n}(x, 1 - x)\) is a polynomial of degree \(\leq m\) in \(x\) and all terms in the power series of \(x^{m+1}(1 - x)^{-n-1}P_{n,m}(1 - x, x)\) have degree \(\geq m + 1\). Hence \(P_{m,n}(x, 1 - x)\) equals the power series of \((1 - x)^{-n-1}\) truncated after the term of \(x^m\), i.e.,

\[
P_{m,n}(x, 1 - x) = \sum_{k=0}^{m} \frac{(n + 1)_k}{k!} x^k.
\]

Then substitution proves (dM), and its homogeneous form

\[
(x + y)^{m+n+1} = y^{n+1} \sum_{k=0}^{m} \frac{(n + 1)_k}{k!} x^k (x + y)^{m-k} + x^{m+1} \sum_{k=0}^{n} \frac{(m + 1)_k}{k!} y^k (x + y)^{n-k}.
\]
Note that, conversely, \((dM)\) implies the equality

\[
\sum_{k=0}^{m} \binom{m+n+1}{k} x^k (1-x)^{m-k} = \sum_{k=0}^{m} \binom{n+k}{k} x^k.
\]
Note that, conversely, \((dM)\) implies the equality

\[
\sum_{k=0}^{m} \binom{m+n+1}{k} x^k (1-x)^{m-k} = \sum_{k=0}^{m} \binom{n+k}{k} x^k.
\]

This is actually a special case of the Pfaff transformation formula

\[
_{2}F_{1}\left(\begin{array}{c}
a, b \\
\overline{c}
\end{array}; x \right) = (1-x)^{-a} _{2}F_{1}\left(\begin{array}{c}
a, c - b \\
\overline{c}
\end{array}; \frac{x}{x-1} \right)
\]

for Gauß hypergeometric series, defined by

\[
_{2}F_{1}\left(\begin{array}{c}
a, b \\
\overline{c}
\end{array}; z \right) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.
\]
The Bezout identity in $\mathbb{K}[x]$ states that if $p, q \in \mathbb{K}[x]$ with $\deg p, \deg q > 0$ and $\gcd(p(x), q(x)) = 1$, then there exist unique $r, s \in \mathbb{K}[x]$ with $\deg r = (\deg q) - 1$ and $\deg s = (\deg p) - 1$ such that $1 = p(x)r(x) + q(x)s(x)$.

Daubechies used this identity to start with (the $m = n$ case of) $1 = (1 - x)^{n+1}r_{m,n}(x) + x^{m+1}s_{m,n}(x)$, where $r_{m,n}(x)$ and $s_{m,n}(x)$ are polynomials of respective degree $m$ and $n$, not yet necessarily explicitly given.

Her further analysis was the same as Chaundy and Bullard's, while she made use of the symmetry $r_{m,n}(x) = s_{n,m}(1-x)$.
The **Bezout identity** in $K[x]$ states that
if $p, q \in K[x]$ with $\deg p, \deg q > 0$ and $\gcd(p(x), q(x)) = 1$,
then there exist unique $r, s \in K[x]$
with $\deg r = (\deg q) - 1$ and $\deg s = (\deg p) - 1$ such that

$$1 = p(x)r(x) + q(x)s(x).$$
Daubechies’ proof

The **Bezout identity** in $K[x]$ states that
if $p, q \in K[x]$ with $\deg p, \deg q > 0$ and $\gcd(p(x), q(x)) = 1$, then there exist unique $r, s \in K[x]$ with $\deg r = (\deg q) - 1$ and $\deg s = (\deg p) - 1$ such that

$$1 = p(x)r(x) + q(x)s(x).$$

Daubechies used this identity to start with (the $m = n$ case of)

$$1 = (1 - x)^{n+1} r_{m,n}(x) + x^{m+1} s_{m,n}(x),$$

where $r_{m,n}(x)$ and $s_{m,n}(x)$ are polynomials of respective degree $m$ and $n$, not yet necessarily explicitly given.
The **Bezout identity** in $K[x]$ states that if $p, q \in K[x]$ with $\deg p, \deg q > 0$ and $\gcd(p(x), q(x)) = 1$, then there exist unique $r, s \in K[x]$ with $\deg r = (\deg q) - 1$ and $\deg s = (\deg p) - 1$ such that

$$1 = p(x)r(x) + q(x)s(x).$$

Daubechies used this identity to start with (the $m = n$ case of)

$$1 = (1 - x)^{n+1}r_{m,n}(x) + x^{m+1}s_{m,n}(x),$$

where $r_{m,n}(x)$ and $s_{m,n}(x)$ are polynomials of respective degree $m$ and $n$, not yet necessarily explicitly given.

Her further analysis was the same as Chaundy and Bullard’s, while she made use of the symmetry $r_{m,n}(x) = s_{n,m}(1 - x)$. 

---

**Generalizations of an identity of de Montmort**
A proof by generating functions

(Damjanovic–Klamkin–Ruehr; Prodinger)
A proof by generating functions

(Damjanovic–Klamkin–Ruehr; Prodinger)

Let

\[ f(u, v; x) := \sum_{m,n \geq 0} u^m v^n (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k. \]
A proof by generating functions

(Damjanovic–Klamkin–Ruehr; Prodinger)

Let

\[ f(u, v; x) := \sum_{m, n \geq 0} u^m v^n (1 - x)^{n+1} \sum_{k=0}^m \binom{n + k}{k} x^k. \]

Then

\[ f(v, u; 1 - x) = \sum_{m, n \geq 0} u^m v^n x^{m+1} \sum_{k=0}^n \binom{m + k}{k} (1 - x)^k. \]
A proof by generating functions

(Damjanovic–Klamkin–Ruehr; Prodinger)

Let

\[ f(u, v; x) := \sum_{m, n \geq 0} u^m v^n (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k. \]

Then

\[ f(v, u; 1 - x) = \sum_{m, n \geq 0} u^m v^n x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1 - x)^k. \]

We have

\[ f(u, v; x) = \frac{1}{1 - u} \sum_{n \geq 0} v^n (1 - x)^{n+1} \sum_{k \geq 0} \binom{n+k}{k} (ux)^k \]

\[ = \frac{1}{1 - u} \sum_{n \geq 0} v^n (1 - x)^{n+1} \frac{1}{(1 - ux)^{n+1}} \]

\[ = \frac{1 - x}{(1 - u)(1 - ux)} \frac{1}{1 - \frac{v(1-x)}{1-ux}} \]

\[ = \frac{1 - x}{1 - u} \frac{1}{1 - ux - v(1-x)}. \]
Hence

\[ f(v, u; 1 - x) = \frac{x}{1 - v} \frac{1}{1 - ux - v(1 - x)} \]

and

\[ f(u, v; x) + f(v, u; 1 - x) = \frac{1}{1 - ux - v(1 - x)} \left( \frac{1-x}{1-u} + \frac{x}{1-v} \right) \]

\[ = \frac{1}{(1-u)(1-v)}. \]
Hence

\[ f(v, u; 1 - x) = \frac{x}{1 - v} \frac{1}{1 - ux - v(1 - x)} \]

and

\[ f(u, v; x) + f(v, u; 1 - x) = \frac{1}{1 - ux - v(1 - x)} \left( \frac{1 - x}{1 - u} + \frac{x}{1 - v} \right) \]

\[ = \frac{1}{(1 - u)(1 - v)}. \]

So

\[ f(u, v; x) + f(v, u; 1 - x) = \sum_{m, n \geq 0} u^m v^n, \]

which yields (dM) by combining the last equation with the first two equations of the previous page, and taking the coefficient of \( u^m v^n \). \( \square \)
Recall that de Montmort's identity is given by

\[ 1 = (1 - x)^{n+1} \sum_{k=0}^{m}(n+k) \frac{x^k}{k!} + x^{m+1} \sum_{k=0}^{n}(m+k) \frac{y^k}{k!}(1 - x)^k. \]

(dM)

The substitution \( x \mapsto \frac{x}{x+y} \) and multiplication of both sides by \( (x+y)^{m+n+2} \) gives the homogeneous form of the identity:

\[ (x+y)^{m+n+2} = y^{n+1} \sum_{k=0}^{m}(n+k) \frac{x^k}{k!}(x+y)^{m+1-k} - x^{m+1} \sum_{k=0}^{n}(m+k) \frac{y^k}{k!}(x+y)^{n+1-k}. \]

(Here, \( x \) and \( y \) are commuting variables.)

The last identity has a combinatorial interpretation in terms of weighted lattice paths.
Recall that \textit{de Montmort's identity} is given by

\[1 = (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n + k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m + k}{k} (1 - x)^k. \quad \text{(dM)}\]
Recall that de Montmort’s identity is given by

\[
1 = (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1 - x)^k. \quad (dM)
\]

The substitution \( x \mapsto x/(x+y) \) and multiplication of both sides by \((x+y)^{m+n+2}\) gives the homogeneous form of the identity:

\[
(x+y)^{m+n+2} = y^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k (x+y)^{m+1-k} + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} y^k (x+y)^{n+1-k}.
\]

(Here, \( x \) and \( y \) are commuting variables.)
Recall that de Montmort’s identity is given by

\[ 1 = (1 - x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1 - x)^k. \]  

(dM)

The substitution \( x \mapsto \frac{x}{(x + y)} \) and multiplication of both sides by \((x + y)^{m+n+2}\) gives the homogeneous form of the identity:

\[ (x + y)^{m+n+2} = y^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k (x + y)^{m+1-k} + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} y^k (x + y)^{n+1-k}. \]

(Here, \( x \) and \( y \) are commuting variables.)

The last identity has a combinatorial interpretation in terms of weighted lattice paths.
Consider all lattice paths from \((0, 0)\) to \((m + 1, n + 1)\) in the planar integer lattice (using only unit east and north steps). Such a path \(P\) consists of \(m + n + 2\) successive unit steps.
Consider all lattice paths from \((0,0)\) to \((m+1,n+1)\) in the planar integer lattice (using only unit east and north steps).
Such a path \(P\) consists of \(m+n+2\) successive unit steps.

The **weight** \(w(P)\) of a path \(P\) is defined to be the product of the weight of the respective steps \(s\) of the path, i.e., \(w(P) = \prod_{s \in P} w(s)\).
Consider all lattice paths from \((0, 0)\) to \((m + 1, n + 1)\) in the planar integer lattice (using only unit east and north steps). Such a path \(P\) consists of \(m + n + 2\) successive unit steps.

The **weight** \(w(P)\) of a path \(P\) is defined to be the product of the weight of the respective steps \(s\) of the path, i.e., \(w(P) = \prod_{s \in P} w(s)\).

Define the weight function \(w\) as follows:

\[
\begin{align*}
    w((i, j) \rightarrow (i + 1, j)) &= \begin{cases} 
        x & (j < n+1), \\
        x + y & (j = n+1), 
    \end{cases} \\
    w((i, j) \rightarrow (i, j + 1)) &= \begin{cases} 
        y & (i < m+1), \\
        x + y & (i = m+1).
    \end{cases}
\end{align*}
\]

It is not difficult to see that the generating function of all paths (from \((0, 0)\) to \((m + 1, n + 1)\)) is

\[
\sum_{P} w(P) = (x + y)^{m+n+2}.
\]
Consider all lattice paths from \((0, 0)\) to \((m + 1, n + 1)\) in the planar integer lattice (using only unit east and north steps). Such a path \(P\) consists of \(m + n + 2\) successive unit steps.

The weight \(w(P)\) of a path \(P\) is defined to be the product of the weight of the respective steps \(s\) of the path, i.e., \(w(P) = \prod_{s \in P} w(s)\). Define the weight function \(w\) as follows:

\[
\begin{align*}
    w((i, j) \rightarrow (i + 1, j)) &:= \begin{cases} 
        x & (j < n + 1), \\
        x + y & (j = n + 1),
    \end{cases} \\
    w((i, j) \rightarrow (i, j + 1)) &:= \begin{cases} 
        y & (i < m + 1), \\
        x + y & (i = m + 1).
    \end{cases}
\end{align*}
\]

It is not difficult to see that the generating function of all paths (from \((0, 0)\) to \((m + 1, n + 1)\)) is

\[
\sum_{P} w(P) = (x + y)^{m+n+2}.
\]
Generalizations of an identity of de Montmort
On the other hand, each path $P$ ends either with a horizontal step or with a vertical step.
On the other hand, each path $P$ ends either with a horizontal step or with a vertical step.

Consider first the paths which end with a horizontal step.

Then the last vertical step will be $(k, n) \to (k, n + 1)$ for some $k \in \{0, 1, \ldots, m\}$.

For given $k$ all such paths have weight $y^{n+1}x^k(x + y)^{m+1-k}$ and the number of such paths is $\binom{n+k}{k}$.
On the other hand, each path \( P \) ends either with a horizontal step or with a vertical step.

Consider first the paths which end with a horizontal step.

Then the last vertical step will be \((k, n) \rightarrow (k, n + 1)\) for some \( k \in \{0, 1, \ldots, m\} \).

For given \( k \) all such paths have weight \( y^{n+1}x^k(x+y)^{m+1-k} \) and the number of such paths is \( \binom{n+k}{k} \).

Hence the sum of the weights of all paths which end with a horizontal step equals

\[
y^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k (x+y)^{m+1-k}.
\]
On the other hand, each path $P$ ends either with a horizontal step or with a vertical step.

Consider first the paths which end with a horizontal step.

Then the last vertical step will be $(k, n) \rightarrow (k, n + 1)$ for some $k \in \{0, 1, \ldots, m\}$.

For given $k$ all such paths have weight $y^{n+1}x^k(x + y)^{m+1-k}$ and the number of such paths is $\binom{n+k}{k}$.

Hence the sum of the weights of all paths which end with a horizontal step equals

$$y^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k(x + y)^{m+1-k}.$$ 

Similarly, the sum of the weights of all paths which end with a vertical step equals

$$x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} y^k(x + y)^{n+1-k}.$$
A proof using the beta integral

By the evaluation of the beta integral we have for $x \in (0, 1)$:

$1 = \frac{(m+n+1)!}{m!n!} \int_0^1 t^m (1-t)^n \, dt = \frac{(m+n+1)!}{m!n!} \int_0^x t^m (1-t)^n \, dt + \frac{(m+n+1)!}{m!n!} \int_x^1 t^m (1-t)^n \, dt$.

Then (dM) will follow from this identity if we can prove that

$\frac{(m+n+1)!}{m!n!} \int_0^x t^m (1-t)^n \, dt = x^{m+1} \sum_{k=0}^{m} \binom{m+1}{k} (1-x)^k$. 

Generalizations of an identity of de Montmort
A proof using the beta integral

By the evaluation of the beta integral we have for \( x \in (0, 1) \):

\[
1 = \frac{(m + n + 1)!}{m! \, n!} \int_0^1 t^m(1 - t)^n \, dt \\
= \frac{(m + n + 1)!}{m! \, n!} \int_0^x t^m(1 - t)^n \, dt + \frac{(m + n + 1)!}{m! \, n!} \int_x^1 t^m(1 - t)^n \, dt \\
= \frac{(m + n + 1)!}{m! \, n!} \int_0^x t^m(1 - t)^n \, dt + \frac{(m + n + 1)!}{m! \, n!} \int_0^{1-x} t^n(1 - t)^m \, dt.
\]
By the evaluation of the *beta integral* we have for \( x \in (0, 1) \):

\[
1 = \frac{(m + n + 1)!}{m! \ n!} \int_{0}^{1} t^{m}(1 - t)^{n} \, dt
\]

\[
= \frac{(m + n + 1)!}{m! \ n!} \int_{0}^{x} t^{m}(1 - t)^{n} \, dt + \frac{(m + n + 1)!}{m! \ n!} \int_{x}^{1} t^{m}(1 - t)^{n} \, dt
\]

\[
= \frac{(m + n + 1)!}{m! \ n!} \int_{0}^{x} t^{m}(1 - t)^{n} \, dt + \frac{(m + n + 1)!}{m! \ n!} \int_{0}^{1-x} t^{n}(1 - t)^{m} \, dt.
\]

Then \((dM)\) will follow from this identity if we can prove that

\[
\frac{(m + n + 1)!}{m! \ n!} \int_{0}^{x} t^{m}(1 - t)^{n} \, dt = x^{m+1} \sum_{k=0}^{n} \frac{(m + 1)_{k}}{k!} (1 - x)^{k}.
\]
The last identity follows by the string of equalities

\[
\int_0^x t^m (1 - t)^n \, dt = x^{m+1} \int_0^1 s^m (1 - s + s(1 - x))^n \, ds \\
= x^{m+1} \sum_{k=0}^{n} \binom{n}{k} (1 - x)^k \int_0^1 s^{m+k} (1 - s)^{n-k} \, ds \\
= \frac{m! \ n! \ x^{m+1}}{(m + n + 1)!} \sum_{k=0}^{n} \frac{(m + 1)_k}{k!} (1 - x)^k.
\]
The last identity follows by the string of equalities

\[
\int_0^x t^m (1 - t)^n \, dt = x^{m+1} \int_0^1 s^m (1 - s + s(1 - x))^n \, ds
\]

\[
= x^{m+1} \sum_{k=0}^n \binom{n}{k} (1 - x)^k \int_0^1 s^{m+k} (1 - s)^{n-k} \, ds
\]

\[
= \frac{m! \, n! \, x^{m+1}}{(m + n + 1)!} \sum_{k=0}^n \frac{(m + 1)_k}{k!} (1 - x)^k.
\]

The last integral on the previous page is an incomplete beta function, which is usually expressed as a hypergeometric function (also for \(m, n\) complex with \(\Re m, \Re n > -1\)):

\[
B_x(m+1, n+1) := \int_0^x t^m (1 - t)^n \, dt = \frac{1}{m + 1} x^{m+1} \, _2F_1\left(-n, m + 1; m + 2; x\right)
\]

\((x \in (0, 1))\). The proof is by binomial expansion of \((1 - t)^n\).
Thus, we have

\[
1 = \frac{\Gamma(n + m + 2)}{\Gamma(m + 1)\Gamma(n + 1)} B_x(m + 1, n + 1) \\
+ \frac{\Gamma(n + m + 2)}{\Gamma(m + 1)\Gamma(n + 1)} B_{1-x}(n + 1, m + 1)
\]

\((x \in (0, 1), \ m, n \in \mathbb{C}, \ \Re m, \Re n > -1)\).

which even extends to complex (non-integer) values of \(m, n\).
Thus, we have

\[ 1 = \frac{\Gamma(n + m + 2)}{\Gamma(m + 1)\Gamma(n + 1)} B_x(m + 1, n + 1) \]
\[ + \frac{\Gamma(n + m + 2)}{\Gamma(m + 1)\Gamma(n + 1)} B_{1-x}(n + 1, m + 1) \]

\[ (x \in (0, 1), \ m, n \in \mathbb{C}, \ \Re m, \Re n > -1) \]  

which even extends to complex (non-integer) values of \( m, n \). 

Thus we have a **complex extension** of \((dM)\). The expression

\[ p_{m,n}(x) = (1 - x)^{n+1} \sum_{k=0}^{m} \frac{\Gamma(n + k + 1)}{\Gamma(n + 1) \Gamma(k + 1)} x^k \]

in de Montmort’s identity extends to

\[ p_{m,n}(x) = \frac{\Gamma(m + n + 2)}{\Gamma(m + 1)\Gamma(n + 1)} B_{1-x}(n + 1, m + 1) \]

\[ (x \in (0, 1), \ m, n \in \mathbb{C}, \ \Re n > -1) \]
Consider (*)& for $x \in (0, 1)$ and $n \in \mathbb{C}$ with $\Re n > -1$, and apply the fractional extension of finite sums proposed by Müller & Schleicher (2005).
Consider $(\ast)$ for $x \in (0, 1)$ and $n \in \mathbb{C}$ with $\Re n > -1$, and apply the fractional extension of finite sums proposed by Müller & Schleicher (2005). Since for $k \in \mathbb{C}$ with $\Re k \geq 0$ we have

$$f(k) := \frac{\Gamma(n + k + 1)}{\Gamma(n + 1) \Gamma(k + 1)} x^k (1 - x)^{n+1} = o(1) \quad \text{as } \Re k \to \infty,$$
Consider \((\ast)\) for \(x \in (0, 1)\) and \(n \in \mathbb{C}\) with \(\Re n > -1\), and apply the fractional extension of finite sums proposed by Müller & Schleicher (2005). Since for \(k \in \mathbb{C}\) with \(\Re k \geq 0\) we have

\[
f(k) := \frac{\Gamma(n + k + 1)}{\Gamma(n + 1) \Gamma(k + 1)} x^k (1 - x)^{n+1} = o(1) \quad \text{as } \Re k \to \infty,
\]

their recipe of fractional extension of the sum \(\sum_{k=0}^{m} f(k)\) is

\[
p_{m,n}(x) = \sum_{k=0}^{\infty} (f(k) - f(k + m + 1)) = \]

\[
(1 - x)^{n+1} \sum_{k=0}^{\infty} \frac{(n + 1)_k}{k!} x^k - (1 - x)^{n+1} \sum_{k=0}^{\infty} \frac{\Gamma(n + m + k + 2)}{\Gamma(n + 1) \Gamma(m + k + 2)} x^{k+m+1}
\]

\[
= 1 - \frac{\Gamma(n + m + 2)}{\Gamma(m + 2) \Gamma(n + 1)} x^{m+1} (1 - x)^{n+1} \, _2F_1\left(\frac{n + m + 2}{m + 2}; x\right)
\]

\[
= 1 - \frac{\Gamma(n + m + 2)}{\Gamma(m + 2) \Gamma(n + 1)} x^{m+1} \, _2F_1\left(-n, m + 1; m + 2; x\right)
\]

\[
= 1 - \frac{\Gamma(n + m + 2)}{\Gamma(m + 1) \Gamma(n + 1)} B_x(m + 1, n + 1),
\]

which gives the same result.
Just as (dM) can be obtained by splitting a beta integral into two parts and evaluating the resulting incomplete beta integrals, we can prove and interpret Damjanovic, Klamkin and Ruehr’s multivariable extension of (dM) by splitting Dirichlet’s \((n − 1)\)-dimensional beta integral with nonnegative integer exponents into \(n\) parts.
Just as \((dM)\) can be obtained by splitting a beta integral into two parts and evaluating the resulting incomplete beta integrals, we can prove and interprete Damjanovic, Klamkin and Ruehr's multivariable extension of \((dM)\) by splitting Dirichlet's \((n - 1)\)-dimensional beta integral with nonnegative integer exponents into \(n\) parts.

For convenience, we state Dirichlet's \(n\)-dimensional beta integral:
Let \(\Delta_n\) be the simplex in \(\mathbb{R}^n\) which has vertices 0 and the standard basis vectors \(e_1, \ldots, e_n\). Let \(a_1, \ldots, a_n, b \in \mathbb{C}\) with real part \(> -1\). Then Dirichlet's integral is as follows.

\[
I_{a_1, \ldots, a_{n+1}} := \int_{\Delta_n} t_1^{a_1} \cdots t_n^{a_n} (1 - t_1 - \cdots - t_n)^{a_{n+1}} dt_1 \cdots dt_n
= \frac{\Gamma(a_1 + 1) \cdots \Gamma(a_{n+1} + 1)}{\Gamma(a_1 + \cdots + a_{n+1} + n + 1)}.
\]

Note that \(I_{a_1, \ldots, a_{n+1}}\) is symmetric in \(a_1, \ldots, a_{n+1}\).
There are various generalizations of \( (dM) \).

The first generalization of \( (dM) \) is a simple \( q \)-analogue:

\[
1 = \left( x; q \right)_n + 1 \sum_{k=0}^{n} \left[ \begin{array}{c} n+k \\ k \end{array} \right] q^k x^k + \left( x; q \right)_m + 1 \sum_{k=0}^{m} \left[ \begin{array}{c} m+k \\ k \end{array} \right] q^k \left( x; q \right)_k.
\]

Here, \( \left( x; q \right)_k \) is the \( q \)-Pochhammer symbol and \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) is the \( q \)-binomial coefficient.

While the first and second term on the right-hand side of \( (dM) \) are evidently symmetric with respect to the simple involution \((x, n, m) \mapsto (1-x, m, n)\)), the corresponding symmetry for \( (q-dM) \) is less evident.

To find the "hidden" symmetry, we look at the transition matrix of the two respective polynomial bases \( x^n \), \( \left( x; q \right)_n \).
There are various generalizations of (dM).
q-Extensions

There are various generalizations of (dM).

The first generalization of (dM) is a simple $q$-analogue:

$$1 = (x; q)_{n+1} \sum_{k=0}^{m} \binom{n+k}{k}_q x^k + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k}_q q^k (x; q)_k. \quad (q\text{-dM})$$

Here, $(x; q)_k := \prod_{j=0}^{k-1} (1 - xq^j)$ is the $q$-Pochhammer symbol and

$$\binom{n}{k}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},$$

where $n \geq k \geq 0$, is the $q$-binomial coefficient.
$q$-Extensions

There are various generalizations of $(dM)$.

The first generalization of $(dM)$ is a simple $q$-analogue:

$$1 = (x; q)_{n+1} \sum_{k=0}^{m} \left[ \begin{array}{c} n+k \\ k \end{array} \right]_q x^k + x^{m+1} \sum_{k=0}^{n} \left[ \begin{array}{c} m+k \\ k \end{array} \right]_q q^k (x; q)_k. \quad (q-dM)$$

Here, $(x; q)_k := \prod_{j=0}^{k-1} (1 - xq^j)$ is the $q$-Pochhammer symbol and $\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$, where $n \geq k \geq 0$, is the $q$-binomial coefficient.

While the first and second term on the right-hand side of $(dM)$ are evidently symmetric with respect to the simple involution $(x, n, m) \mapsto (1 - x, m, n)$, the corresponding symmetry for $(q-dM)$ is less evident.
There are various generalizations of \((dM)\).

The first generalization of \((dM)\) is a simple \(q\)-analogue:

\[
1 = (x; q)_{n+1} \sum_{k=0}^{m} \left[ \begin{array}{c} n + k \\ k \end{array} \right]_q x^k + x^{m+1} \sum_{k=0}^{n} \left[ \begin{array}{c} m + k \\ k \end{array} \right]_q q^k (x; q)_k. \quad (q\text{-dM})
\]

Here, \((x; q)_k := \prod_{j=0}^{k-1} (1 - xq^j)\) is the \(q\)-Pochhammer symbol and \nabla^{\left[\begin{array}{c} n \\ k \end{array}\right]}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \) where \(n \geq k \geq 0\), is the \(q\)-binomial coefficient.

While the first and second term on the right-hand side of \((dM)\) are evidently symmetric with respect to the simple involution \((x, n, m) \mapsto (1 - x, m, n)\), the corresponding symmetry for \((q\text{-dM})\) is less evident.

To find the “hidden” symmetry, we look at the transition matrix of the two respective polynomial bases \([x^n]_{n \geq 0}, [(x; q)_n]_{n \geq 0} \).
By the \( q \)-binomial theorem, we have

\[
(x; q)_n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{k(k-1)/2} x^k.
\]
By the $q$-binomial theorem, we have

$$(x; q)_n = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{{k \choose 2}} x^k.$$ 

Now, defining the (lower-triangular) matrix $F = (f_{nk})_{n,k \in \mathbb{Z}}$ by its entries

$$f_{nk} = \binom{n}{k}_q (-1)^k q^{{k \choose 2}},$$

the inverse of $F$ is known to be the (lower-triangular) matrix $G = (g_{nk})_{n,k \in \mathbb{Z}}$, with entries

$$g_{nk} = \binom{n}{k}_q (-1)^k q^{{k \choose 2} + k(1-n)}.$$
By the $q$-binomial theorem, we have

$$(x; q)_n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{k(k-1)} x^k.$$  

Now, defining the (lower-triangular) matrix $F = (f_{nk})_{n,k \in \mathbb{Z}}$ by its entries

$$f_{nk} = \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{k(k-1)},$$

the inverse of $F$ is known to be the (lower-triangular) matrix $G = (g_{nk})_{n,k \in \mathbb{Z}}$, with entries

$$g_{nk} = \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{k(k-1)+k(1-n)}.$$  

A simple computation reveals that

$$g_{nk}(q) = f_{nk}(q^{-1}).$$
Therefore, the relation

\[(x; q)_n = \sum_{k=0}^{n} f_{nk}(q)x^k\]

is equivalent to

\[x^n = \sum_{k=0}^{n} f_{nk}(q^{-1})(x; q)_k.\]
Therefore, the relation

\[ (x; q)_n = \sum_{k=0}^{n} f_{nk}(q)x^k \]

is equivalent to

\[ x^n = \sum_{k=0}^{n} f_{nk}(q^{-1})(x; q)_k. \]

It is now natural to define the following linear operator \( \phi \) on \( \mathcal{P} = \mathbb{F}(q)[x] \) (the vector space of polynomials in \( x \) with coefficients that are rational functions in \( q \) over the field \( \mathbb{F} \)) by

\[ \phi \sum_{k \geq 0} c_k(q)x^k = \sum_{k \geq 0} c_k(q^{-1})(x; q)_k. \]
Therefore, the relation

\[(x; q)_n = \sum_{k=0}^{n} f_{nk}(q)x^k\]

is equivalent to

\[x^n = \sum_{k=0}^{n} f_{nk}(q^{-1})(x; q)_k.\]

It is now natural to **define** the following linear operator \( \phi \) on \( P = \mathbb{F}(q)[x] \) (the vector space of polynomials in \( x \) with coefficients that are rational functions in \( q \) over the field \( \mathbb{F} \)) by

\[\phi \sum_{k \geq 0} c_k(q)x^k = \sum_{k \geq 0} c_k(q^{-1})(x; q)_k.\]

Note that \( \phi \) is an **involution** (this follows immediately from the inverse relations) but not a homomorphism (unless \( q = 1 \)).
To derive \((q\text{-dM})\) using Bézout’s identity, observe that for each \(m, n\) there exist unique polynomials \(r_{m,n}\) and \(s_{m,n}\) of degree \(m, n\), respectively, such that

\[
1 = (x; q)_{n+1}r_{m,n}(x; q) + x^{m+1}s_{m,n}(x; q).
\]
To derive \((q\text{-dM})\) using Bézout’s identity, observe that for each \(m, n\) there exist unique polynomials \(r_{m,n}\) and \(s_{m,n}\) of degree \(m, n\), respectively, such that

\[
1 = (x; q)_{n+1} r_{m,n}(x; q) + x^{m+1} s_{m,n}(x; q).
\]

This implies

\[
\frac{1}{(x; q)_{n+1}} = r_{m,n}(x; q) + O(x^{m+1}) \quad \text{as} \quad x \to 0.
\]
To derive \((q\text{-dM})\) using Bézout’s identity, observe that for each \(m, n\) there exist unique polynomials \(r_{m,n}\) and \(s_{m,n}\) of degree \(m, n\), respectively, such that

\[
1 = (x; q)_{n+1}r_{m,n}(x; q) + x^{m+1}s_{m,n}(x; q) .
\]

This implies

\[
\frac{1}{(x; q)_{n+1}} = r_{m,n}(x; q) + O(x^{m+1}) \quad \text{as} \quad x \to 0.
\]

The next step is to compute \(r_{m,n}(x; q)\) by using

\[
\frac{1}{(x; q)_{n+1}} = \sum_{k=0}^{m} \binom{n+k}{k}_q x^k + O(x^{m+1}) \quad \text{as} \quad x \to 0,
\]

a result which is easily deduced from the nonterminating \(q\)-binomial theorem.
If we can show that

$$\phi\left( (x; q)_{n+1} \sum_{k=0}^{m} \left[ \begin{array}{c} n + k \\ k \end{array} \right]_q x^k \right) = x^{n+1} s_{n,m}(x; q),$$

for

$$s_{m,n}(x; q) = \sum_{k=0}^{n} \left[ \begin{array}{c} m + k \\ k \end{array} \right]_q q^k(x; q)_k,$$

then we are done, as $s_{n,m}(x; q)$ must have degree $m$ and be unique.
The computations are as follows:

\[
\phi \left( (x; q)_{n+1} \sum_{k=0}^{m} \binom{n+k}{k}_q x^k \right)
\]

\[
= \phi \sum_{l=0}^{n+1} \sum_{k=0}^{m} \binom{n+1}{l}_q q^{l(2)} \binom{n+k}{k}_q x^{k+l}
\]

\[
= \sum_{l=0}^{n+1} \sum_{k=0}^{m} \binom{n+1}{l}_q q^{l(2)} \binom{n+k}{k}_q^{-1} (x; q)_{k+l}
\]

\[
= \sum_{k=0}^{m} \binom{n+k}{k}_q q^{-1} (x; q)_k \sum_{l=0}^{n+1} \binom{n+1}{l}_q q^{l(2)} (xq^k; q)_l
\]

\[
= \sum_{k=0}^{m} \binom{n+k}{k}_q q^{-1} (x; q)_k x^{n+1} q^{k(n+1)} = x^{n+1} \sum_{k=0}^{m} \binom{n+k}{k}_q q^{k} (x; q)_k,
\]

which settles \((q\text{-dM})\).
We have the following **symmetric** generalization of \((q\text{-dM})\):

\[
1 = (bz, \frac{b}{z}; q)_{n+1} (\frac{b}{a}; q)_{n+1} \sum_{k=0}^{m} (q^{n+1}, az, \frac{a}{z}; q)_k (q, \frac{aq}{b}; q)_k q^k + (az, \frac{a}{z}; q)_{m+1} (\frac{ab}{a}; q)_{m+1} \sum_{k=0}^{n} (q^{m+1}, bz, \frac{b}{z}; q)_k (q, \frac{bq}{a}; q)_k q^k.
\]

This generalizes further to

\[
1 = (bz, \frac{b}{z}; q)_{n+1} (ab, \frac{b}{a}; q)_{n+1} \sum_{k=0}^{m} (q^{n+1}, az, \frac{a}{z}; q)_k (q, \frac{aq}{b}, \frac{abq}{1}; q)_k q^k + (az, \frac{a}{z}; q)_{m+1} (\frac{ab}{a}; q)_{m+1} \sum_{k=0}^{n} (q^{m+1}, bz, \frac{b}{z}; q)_k (q, \frac{bq}{a}, \frac{abq}{1}; q)_k q^k.
\]

And yet further to

\[
1 = (ac, \frac{c}{a}, \frac{bz}{b}; q)_{n+1} (ab, \frac{b}{a}, \frac{cz}{c}; q)_{n+1} \sum_{k=0}^{m} (1 - acq^{n+2k}) (acq^{n+2k}, bcq^{n+2k}, \frac{c}{b}, q^{n+1}, az, \frac{a}{z}; q)_k (1 - acq^{n+2k}) (q, \frac{aq}{b}, \frac{abq}{1}, ac, cq^{n+1+2k}, \frac{czq^{n+1}}{1}; q)_k q^k + (bc, \frac{c}{b}, \frac{az}{a}, \frac{a}{z}; q)_{m+1} (ab, \frac{a}{b}, \frac{cz}{c}, \frac{c}{z}; q)_{m+1} \sum_{k=0}^{n} (1 - bcq^{m+2k}) (bcq^{m+2k}, acq^{m+2k}, \frac{c}{a}, q^{m+1}, bz, \frac{b}{z}; q)_k (1 - bcq^{m+2k}) (q, \frac{bq}{a}, \frac{abq}{1}, bc, cq^{m+1+2k}, \frac{czq^{m+1}}{1}; q)_k q^k.
\]
We have the following symmetric generalization of \((q\text{-dM})\):

\[
1 = \frac{(bz; q)_{n+1}}{(b/a; q)_{n+1}} \sum_{k=0}^{m} \frac{(q^{n+1}, az; q)_k}{(q, aq/b; q)_k} q^k + \frac{(az; q)_{m+1}}{(a/b; q)_{m+1}} \sum_{k=0}^{n} \frac{(q^{m+1}, bz; q)_k}{(q, bq/a; q)_k} q^k
\]

(where \(b\) is a redundant parameter kept for symmetry).
We have the following symmetric generalization of \((q\text{-d}M)\):

\[
1 = \frac{(bz; q)_{n+1}}{(b/a; q)_{n+1}} \sum_{k=0}^{m} \frac{(q^{n+1}, az; q)_k}{(q, aq/b; q)_k} q^k + \frac{(az; q)_{m+1}}{(a/b; q)_{m+1}} \sum_{k=0}^{n} \frac{(q^{m+1}, bz; q)_k}{(q, bq/a; q)_k} q^k
\]

(\text{where } b \text{ is a redundant parameter kept for symmetry}).

This generalizes further to

\[
1 = \frac{(bz, b/z; q)_{n+1}}{(ab, b/a; q)_{n+1}} \sum_{k=0}^{m} \frac{(q^{n+1}, az, a/z; q)_k}{(q, aq/b, abq^{1+n}; q)_k} q^k
\]

\[
+ \frac{(az, a/z; q)_{m+1}}{(ab, a/b; q)_{m+1}} \sum_{k=0}^{n} \frac{(q^{m+1}, bz, b/z; q)_k}{(q, bq/a, abq^{1+m}; q)_k} q^k,
\]
We have the following symmetric generalization of \((q\text{-dM})\):

\[
1 = \left( \frac{b}{a}; q \right)_n + \sum_{k=0}^{m} \left( \frac{q^{n+1}, az; q}{(q, aq/b; q)_k} \right) q^k + \left( \frac{az; q}{(a/b; q)_m} \right) \sum_{k=0}^{n} \left( \frac{q^{m+1}, b; q}{(q, bq/a; q)_k} \right) q^k
\]

(where \(b \) is a redundant parameter kept for symmetry).

This generalizes further to

\[
1 = \left( \frac{b}{a}; q \right)_n + \sum_{k=0}^{m} \left( \frac{q^{n+1}, az, a/z; q}{(q, aq/b, abq^{1+n}; q)_k} \right) q^k
\]

\[
+ \left( \frac{az, a/z; q}{(a/b; q)_m} \right) \sum_{k=0}^{n} \left( \frac{q^{m+1}, b, b/z; q}{(q, bq/a, abq^{1+m}; q)_k} \right) q^k,
\]

and yet further to

\[
1 = \left( \frac{a}{c}, c; q \right)_n + \sum_{k=0}^{m} \frac{(1 - acq^{n+2k})(acq^n, bcq^n, c/b, q^{n+1}, az, a/z; q)_k}{(1 - acq^n)(q, aq/b, abq^{1+n}, ac, cq^{n+1}/z, czq^{n+1}; q)_k} q^k
\]

\[
+ \left( \frac{bc, c/b; q}{(a/b; q)_m} \right) \sum_{k=0}^{n} \frac{(1 - bcq^{m+2k})(bcq^m, acq^m, c/a, q^{m+1}, bz, b/z; q)_k}{(1 - bcq^m)(q, bq/a, abq^{1+m}, bc, cq^{m+1}/z, czq^{m+1}; q)_k} q^k.
\]