Asynchronous Randomized Automata:

How does randomness affect decentralized algorithms execution?

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Asynchronous decentralized algorithms

Most of the real systems are asynchronous:

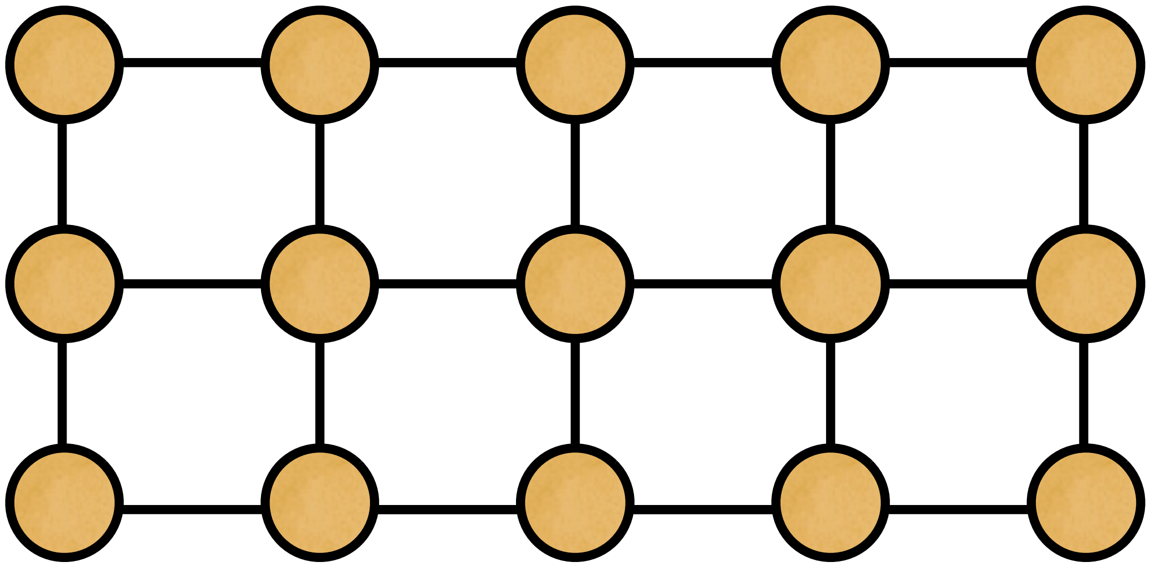
- Networks, physical particles, biological cells...
- How does **randomness introduced by asynchonicity** affect the global behavior of algorithms on these systems?

The kind of decentralized algorithms we consider

Consider a ring network on a ring:

- where each node has two states:
  "has a token" or "does not have a token"
- running an algorithm that redistributes the tokens according to some rules/constraints:
  "I get a token if none of my neighbors have one" or
  "I get a token if my right neighbor has one",...
- Example of question: *How long does it take to reach a "stable configuration"?*

These are cellular automata



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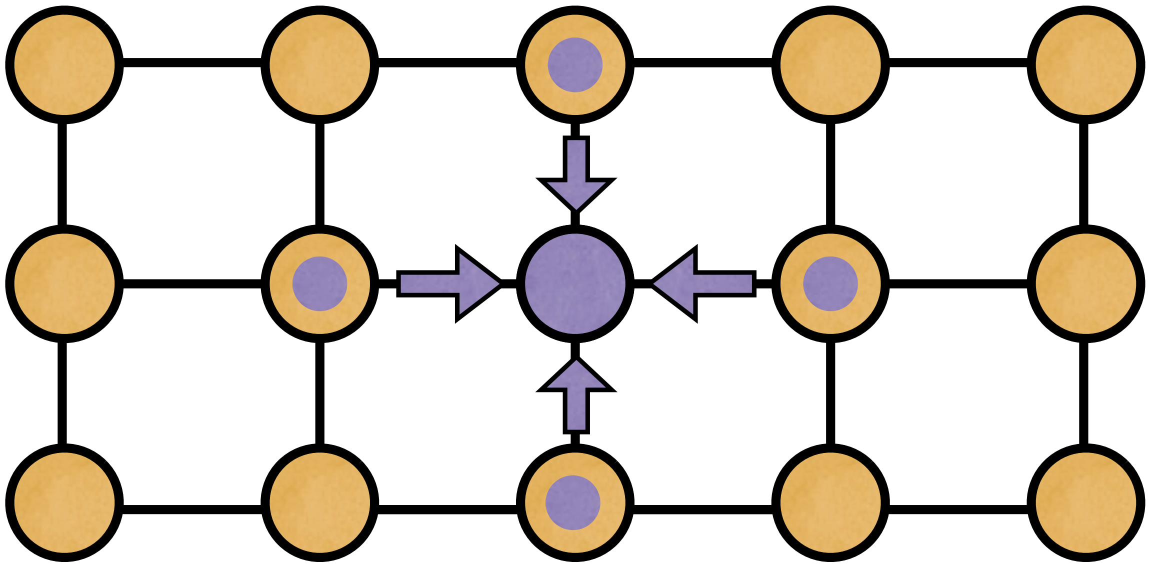
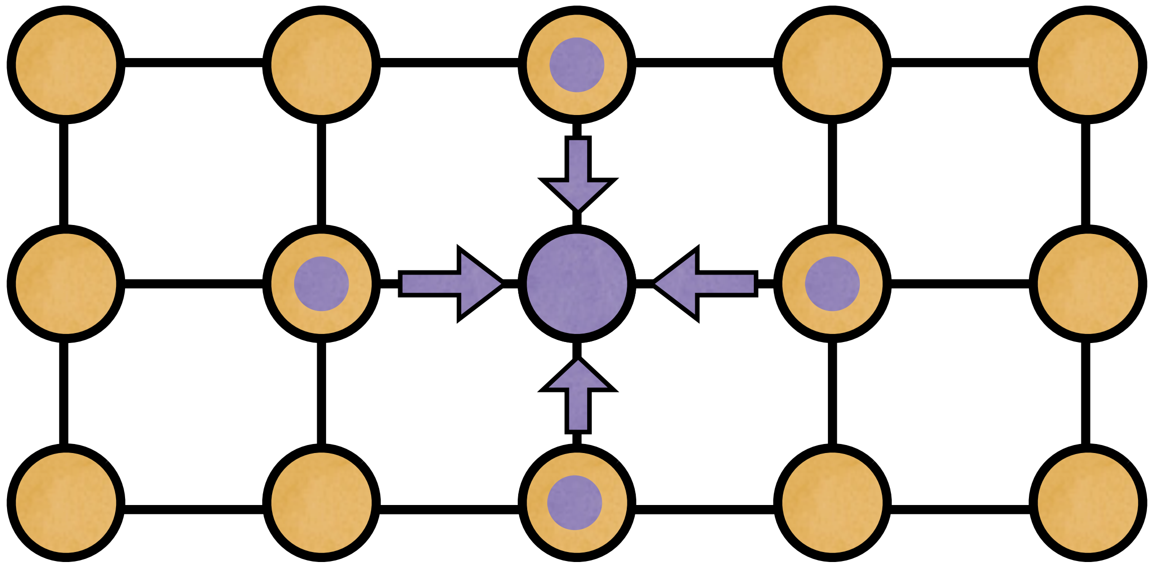


At each time step, each cell updates its states according to the state of its neighbors

These are cellular automata



Extensively used in physics, biology,...

Which one are robust to asynchronism?

Elementary Cellular Automata

We will consider:

- Elementary Cellular Automata : state set = $\{\mathbf{0},\mathbf{1}\}$
- An ECA is defined by its 2^3 transitions
- $\mathbf{2^{(2^3)} = 256}$ ECAs

Asynchronous dynamics

- ***full asynchronism :***
  A daemon chooses a random cell uniformly and updates it
- ***partial asynchronism:***
  Each cell is independently updated with probability $0 < \alpha < 1$

Full synchronism: $\alpha = 1$

Full asynchronism: « limit » for $\alpha \to 0$

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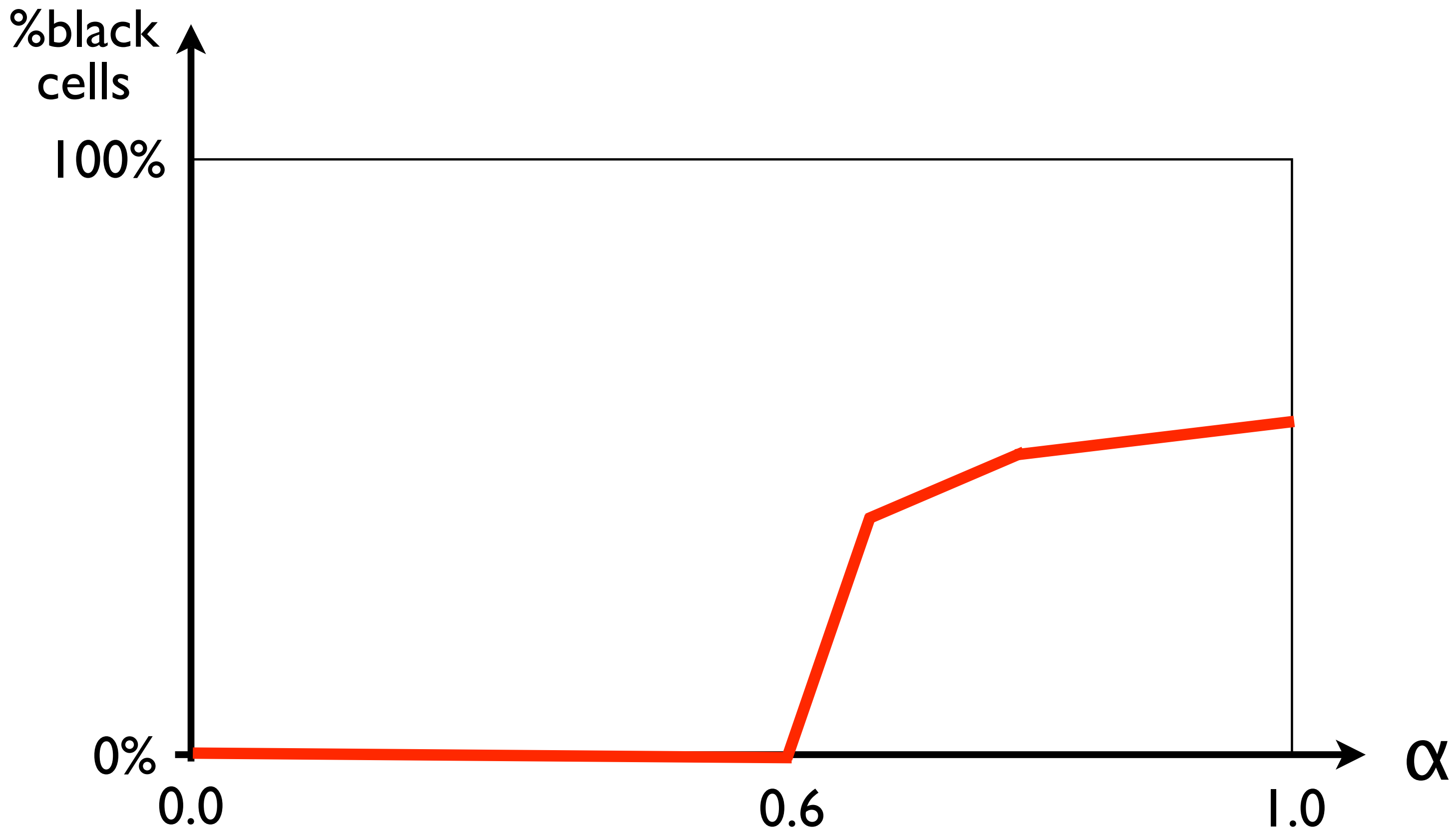
Let’s do some experiments

Example of phase transition for ECA 146



Previous results

- **Theoretical:**
  Gacs (2003): *indecidability of independance to update history*
  Louis (2002): *existence of stationary distribution for randomized cellular automata*
- **Experimental:**
  Buvel & Ingerson (1984): *case study of some 2D auto.*
  Bersini & Detour (1994): *case study of some 2D auto.*
  Schönfish & de Roos (1999): *synchronous vs asynchronous updating, case study*

Stable Cellular Automata
(doublement quiescent CA)

- ***double-quiescent CA (DQECA) :***

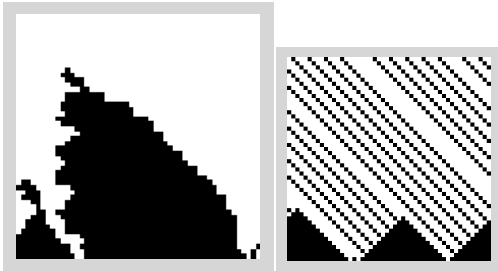
0	0	0	→	0

1	1	1	→	1

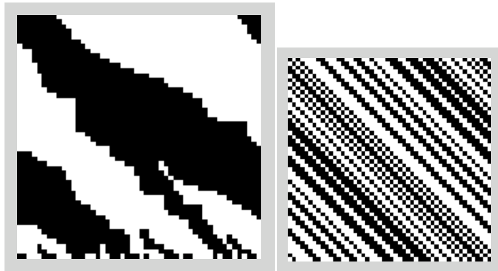
- ***64 DQECAs :***
  only 24 ACEDQs after symmetries

Asynchronous *vs* synchronous dynamics

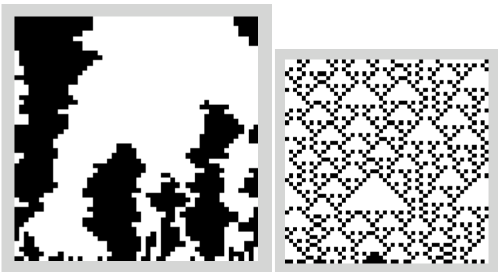
(a) DE 232



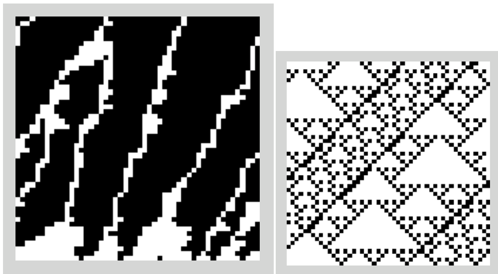
(b) DEFG 130



(c) BDEG 170



(e) BCEFG 146



(e) BCEF 210

(f) BCFG 150

Main theorem

Among the 64 ACEDQs :

- ***55 converge a.s.*** from all initial configuration.
- ***9 diverge a.s.*** from all *(non fixed point)* initial configuration.
- ***worst expected convergence time:***

$$\max_{x \in \{0,1\}^{\mathbb{Z}/n\mathbb{Z}}} \mathbb{E}[\text{Convergence time from } x]$$

only possible values are:

$0,\ \Theta(n \ln n),\ \Theta(n^2),\ \Theta(n^3)$, et $\Theta(n2^n)$

Active transition & fixed points

- ***active transitions:***

A	0	0	0	→	1	**H**	1	1	1	→	0
B	0	0	1	→	1	**G**	1	1	0	→	0
C	1	0	0	→	1	**F**	0	1	1	→	0
D	1	0	1	→	1	**E**	0	1	0	→	0

- ***fixed point:***
  A configuration is a *fixed point* if no update can change it

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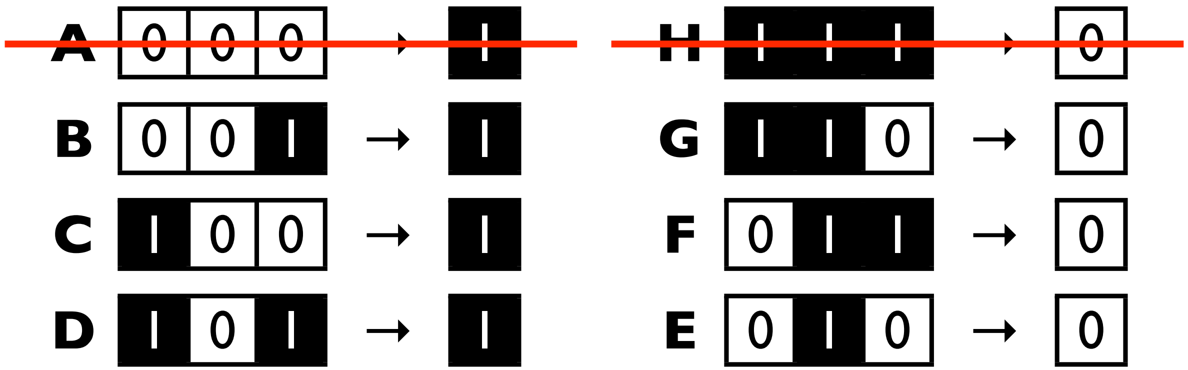

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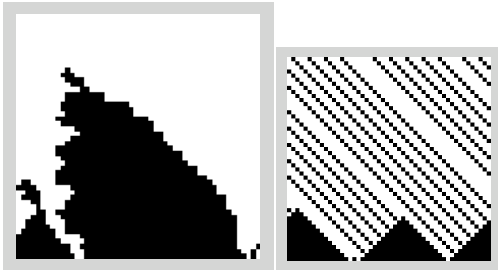

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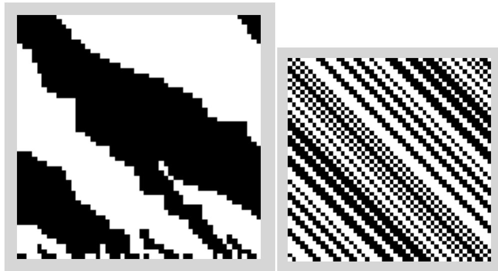

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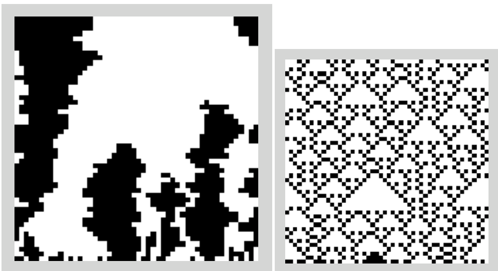

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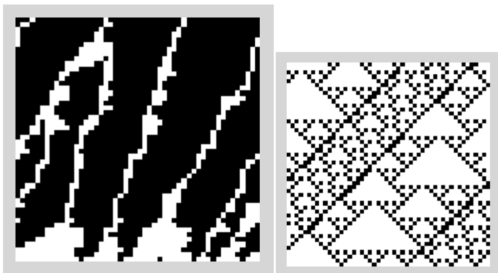

(e) BCEF 210

(f) BCFG 150

Facts

Let x^t denote the configuration at time t

Fact 1 : The number of regions $Z(x^t)$ decreases

Fact 2 : If x is a fixed point of an DQECA,

- if **B** or **C**, then all **0** in x are isolated
- if **F** or **G**, then all **1** in x are isolated
- if **D**, then none of the **0**s in x are isolated
- if **E**, alors none of the **1**s in x are isolated

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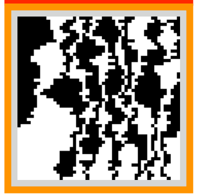
232


130


170


210


150

Behavior	ACE (#)	Rule	01	10	010	101	Convergence
Identity	204 (1)	∅	•	•	•	•	0
Coupon collector	200 (2)	E	•	•	″	•	$\Theta(n \ln n)$
	232 (1)	DE	•	•	″	″	
Monotone	206 (4)	B	←	•	•	•	$\Theta(n^2)$
	222 (2)	BC	←	→	•	•	
	234 (4)	BDE	←	•	″	″	
	250 (2)	BCDE	←	→	″	″	
	202 (4)	BE	←	•	″	•	
	192 (4)	EF	→	•	″	•	
	218 (2)	BCE	←	→	″	•	
	128 (2)	EFG	→	←	″	•	
Baised random walks	242 (4)	BCDEF	⟷	→	″	″	
	130 (4)	BEFG	⟷	←	″	•	
Random walks	226 (2)	BDEF	⟷	•	″	″	$\Theta(n^3)$
	170 (2)	BDEG	←	←	″	″	
	178 (1)	BCDEFG	⟷	⟷	″	″	
	194 (4)	BEF	⟷	•	″	•	
	138 (4)	BEG	←	←	″	•	
	146 (2)	BCEFG	⟷	⟷	″	•	
Biased random walk	210 (4)	BCEF	⟷	→	″	•	$\Theta(n2^n)$
Diverging	198 (2)	BF	⟷	•	•	•	Diverging
	142 (2)	BG	←	←	•	•	
	214 (4)	BCF	⟷	→	•	•	
	150 (1)	BCFG	⟷	⟷	•	•	

Sketch of the analysis

- ***coupon collector:*** direct proof
- ***other converging :***
  Coupling with a proper *variant* (Lyapunov function) based on random walks or martingales
- ***diverging:*** fixed points are unreachable

Two examples:

a quadractic DQECA

an exponential DQECA

Monotonic quadratic: BC 222

The number of **0**s decreases at each active update:

B [0|0|1] → [1]   **C** [1|0|0] → [1]

- *If x^t is not a fixed point*, the expected variation of $X_t = |x^t|_0$ is ≤ $-1/n$
- *If $X_t = 0$*, the process has converged

⇒ *BC converges a.s. and its worst expected convergence time is* $\Theta(n^2)$

Probabilistic lemmas (1)

- ***Coupling the a random process X_t,*** *monotonic* ou *random walk* (martingale) on a finite domain, such that the convergence of X_t *implies the convergence of the automata*
- For **BC**, $X_t = |x^t|_0$
- Another example: for **BEFG**, $X_t = |x^t|_0 + Z(x^t)$

Probabilistic lemmas (2)

Essentially three types of random process X_t :

- Monotonic and biaised random walks towards the fixed point:

  $$\mathbb{E}[\Delta X_{t+1} \,|\, X_t] \leq -1/n$$
  $$\text{Worst convergence time} = \Theta(n^2)$$

- Unbiased random walks (bouncing or not on one of the boundaries) :

  $$\mathbb{E}[\Delta X_{t+1} \,|\, X_t] = 0$$
  $$\Pr\{\Delta X_{t+1} \geq 1\} = \Pr\{\Delta X_{t+1} \leq -1\} \geq 1/n$$
  $$\text{Worst convergence time} = \Theta(n^3)$$

- Biased random walks running away from the fixed point :

  $$\Pr\{\Delta X_{t+1} = 1\} = 2\Pr\{\Delta X_{t+1} = -1\} \geq 1/n$$
  $$\text{Worst convergence time} = \Theta(n2^n)$$

Exponential BCEF 210

B 0 0 1 → 1 **F** 0 1 1 → 0

C 1 0 0 → 1 **E** 0 1 0 → 0

$$X_t = |x^t|_1$$

One 1-région & one 0-région case:

$$\text{If } X_t \leq n-2: \quad \Pr\{\Delta X_{t+1} = 1\} = 2\Pr\{\Delta X_{t+1} = -1\} = 2/n$$

$$\text{If } X_t = n-1: \quad \Pr\{\Delta X_{t+1} = -1\} = 1/n$$

Converges if $X_t = 0$

⇒ *Expected convergence time is* $\mathrm{WCT} = \Theta(n2^n)$

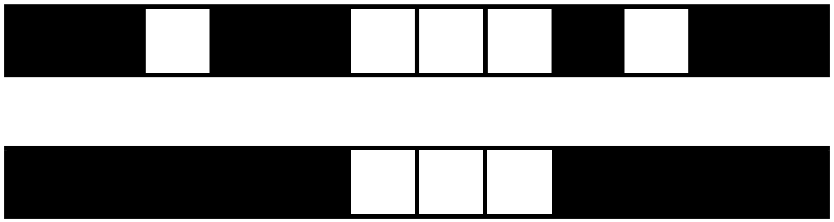
Exponential BCEF 210

B 0 0 1 → 1 F 0 1 1 → 0

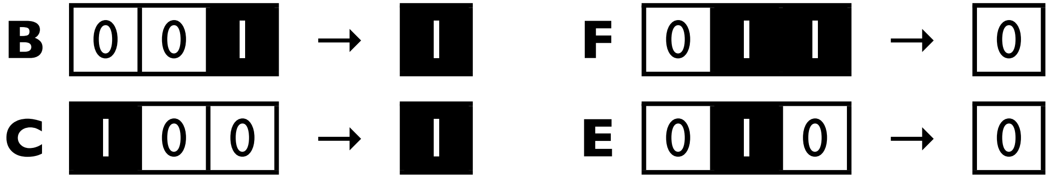
C 1 0 0 → 1 E 0 1 0 → 0

$$X_t = |x^t|_1$$

- ***General case:*** coupling & realignment



Exponential BCEF 210



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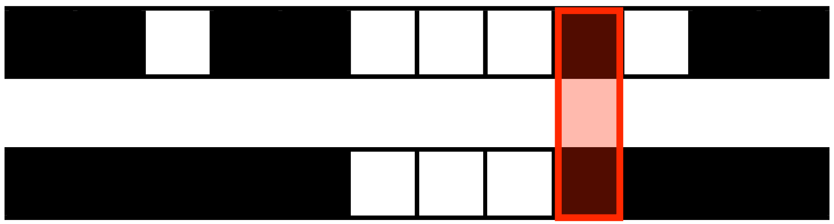

Exponential
BCEF 210

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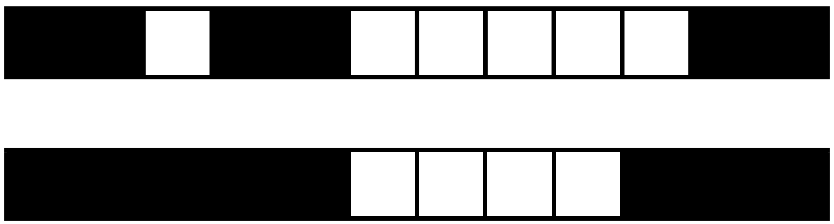

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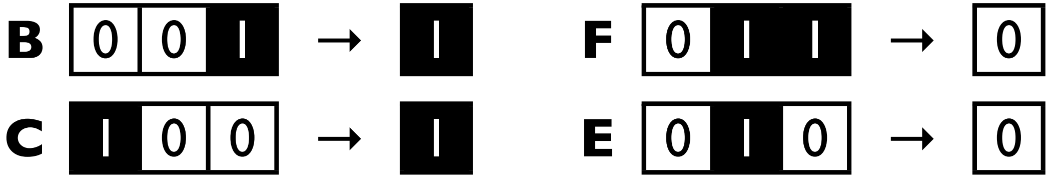

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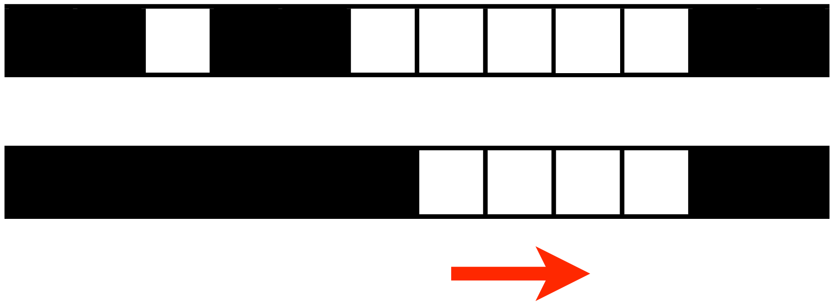


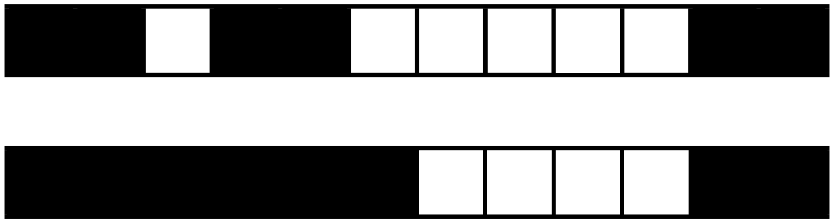
Un exponentiel BCEF 210

B 0 0 1 → 1 **F** 0 1 1 → 0

C 1 0 0 → 1 **E** 0 1 0 → 0

$$X_t = |x^t|_1$$

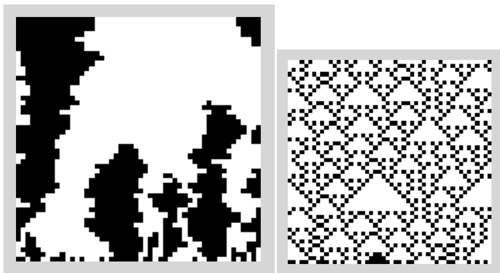
- ***General case:*** coupling & realignment



⇒ *Worst convergence time:* $\mathrm{WCT} = \Theta(n2^n)$

Conclusion

- ***As opposed to synchronous dynamics,*** the asynchronous behavior can be fully determined
- ***Tight values for the worst expected convergence times***



Ongoing work & perspectives

- ***partial asynchronism***
  (D. Regnault)
  Phase transitions on convergence time appear (polynomial / exponentiel)
- **other classes of automata :**
  neighborhood, dimension (realistic models)
- ***Phase transition*** *on convergence time*

Thank you