# Constructing Labeling Schemes through Universal Matrices

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**Abstract.** Let $f$ be a function on pairs of vertices. An *$f$-labeling scheme* for a family of graphs $\mathcal{F}$ labels the vertices of all graphs in $\mathcal{F}$ such that for every graph $G \in \mathcal{F}$ and every two vertices $u, v \in G$, $f(u, v)$ can be inferred by merely inspecting the labels of $u$ and $v$. The *size* of a labeling scheme is the maximum number of bits used in a label of any vertex in any graph in $\mathcal{F}$. This paper illustrates that the notion of universal matrices can be used to efficiently construct $f$-labeling schemes.
Let $\mathcal{F}(n)$ be a family of connected graphs of size at most $n$ and let $\mathcal{C}(\mathcal{F}, n)$ denote the collection of graphs of size at most $n$, such that each graph in $\mathcal{C}(\mathcal{F}, n)$ is composed of a disjoint union of some graphs in $\mathcal{F}(n)$. We first investigate methods for translating $f$-labeling schemes for $\mathcal{F}(n)$ to $f$-labeling schemes for $\mathcal{C}(\mathcal{F}, n)$. In particular, we show that in many cases, given an $f$-labeling scheme of size $g(n)$ for a graph family $\mathcal{F}(n)$, one can construct an $f$-labeling scheme of size $g(n) + \log\log n + O(1)$ for $\mathcal{C}(\mathcal{F}, n)$. We also show that in several cases, the above mentioned extra additive term of $\log\log n + O(1)$ is necessary. In addition, we show that the family of $n$-node graphs which are unions of disjoint circles enjoys an adjacency labeling scheme of size $\log n + O(1)$. This illustrates a non-trivial example showing that the above mentioned extra additive term is sometimes not necessary.
We then turn to investigate distance labeling schemes on the class of circles of at most $n$ vertices and show an upper bound of $1.5 \log n + O(1)$ and a lower bound of $4/3 \log n - O(1)$ for the size of any such labeling scheme.
**Keywords:** Labeling schemes, Universal graphs, Universal matrices.

## 1 Introduction

*Motivation and related work* In the fields of communication networks and distributed computing, network representation schemes have been studied extensively. This paper studies a type of representation based on assigning *informative*

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*labels* to the vertices of the network. In most traditional network representations, the names or identifiers given to the vertices betray nothing about the network's structure. In contrast, the labeling schemes studied here involve using more informative and localized labels for the network vertices. The idea is to associate with each vertex a label selected in a such way, that will allow us to infer information about any two vertices *directly* from their labels, without using *any* additional information sources. Hence in essence, this method bases the entire representation on the set of labels alone.

Obviously, without restricting the label size one can encode any desired information, including in particular, the entire graph structure. Our focus is thus on informative labeling schemes using *short* labels. Labeling schemes were previously developed for different graph families and for a variety information types, including adjacency [16, 5], distance [27, 21, 13, 12, 10, 17, 30, 7, 1], tree routing [9, 31], flow and vertex connectivity [20, 15], tree ancestry [4, 3, 17, 2, 18, 19], nearest common ancestor in trees [28, 2] and various other tree functions. See [11] for a survey on (static) labeling schemes. The dynamic version was studied in [23, 22, 8, 14].

The *size* of a graph is the number of vertices in it. The *size* of a labeling scheme for a family $\mathcal{F}$ of graphs is defined as the maximum number of bits assigned in a label of any vertex in any graph in $\mathcal{F}$. The following is an example (see [16]) of an adjacency labeling scheme on the family of $n$-node forests.

**Example:** Given an $n$-node forest, first assign each vertex $v$ a disjoint identifier $id(v)$ in the range $1, 2, \cdots, n$. Then assign each non-root vertex $v$, the label $L(v) = \langle id(v), id(p(v))\rangle$, where $p(v)$ is $v$'s parent in the corresponding tree, and assign each root $r$ the label $L(r) = \langle id(r)\rangle$. Note that two nodes in a forest are neighbors iff one is the parent of the other. Therefore, given the labels of two nodes, one can easily determine whether these nodes are neighbors or not. Clearly, the size of this labeling scheme is $2\log n$. ▮

As shown in [16], the notion of adjacency labeling schemes is strongly related to the notion of vertex induced *universal graphs*. Given a graph family $\mathcal{F}$, a graph $\mathcal{U}$ is $\mathcal{F}$-induced universal if every graph in $\mathcal{F}$ is a vertex induced subgraph of $\mathcal{U}$. In the early 60's, induced universal graphs were studied in [29] for infinite graph families. Induced universal graphs for the family of all $n$-node graphs were studied in [26], for trees, forests, bounded arboricity graphs and planar graphs in [6, 16, 4], for hereditary graphs in [24] and for several families of bipartite graphs in [25].

As proved in [16], a graph family $\mathcal{F}$ (of $n$-node graphs) has an adjacency labeling scheme of size $g(n)$ iff there exists an $\mathcal{F}$-induced universal graph of size $2^{g(n)}$. Therefore, Example 1 implies the existence of an $n^2$-node induced universal graph for the family of $n$-node forests. This bound was further improved in [5] to $2^{\log n + O(\log^* n)}$ by constructing an adjacency labeling scheme for forests with label size $\log n + O(\log^* n)$.

An extension of the notion of an $\mathcal{F}$-induced universal graph was given in [10]. An *$\mathcal{F}$-universal distance matrix* is a square matrix $\mathcal{U}$ containing the distance matrix of every graph in $\mathcal{F}$ as an induced sub-matrix. It was shown in [10] that a graph family $\mathcal{F}$ has a distance labeling scheme of size $g(n)$ iff there exists an $\mathcal{F}$-universal distance matrix with dimension $2^{g(n)}$. We note that to the best of our knowledge, despite the above mentioned relation between labeling schemes and universal distance matrices, no attempt has been made so far to construct labeling schemes based on this relation.

*Our results* We first notice that the notion of a *universal distance matrix* can be generalized into a *universal $f$-matrix* for any type of function $f$ on pairs of vertices. This paper investigates this notion of universal $f$-matrices for various functions and graph families and uses it to explicitly construct upper bounds and lower bounds on the sizes of the corresponding $f$-labeling schemes. To the best of our knowledge, this is the first attempt to explicitly construct labeling schemes based on such notions.

Let $\mathcal{F}(n)$ be a family of connected graphs of size at most $n$ and let $\mathcal{C}(\mathcal{F}, n)$ denote the collection of graphs of size at most $n$, such that each graph in $\mathcal{C}(\mathcal{F}, n)$ is composed of a disjoint union of some graphs in $\mathcal{F}(n)$. We first investigate methods for translating $f$-labeling schemes for $\mathcal{F}(n)$ to $f$-labeling schemes for $\mathcal{C}(\mathcal{F}, n)$. In particular, using the notion of universal $f$-matrices we show that in many cases, given an $f$-labeling scheme of size $g(n)$ for a graph family $\mathcal{F}(n)$, one can construct an $f$-labeling scheme of size $g(n) + \log\log n + O(1)$ for $\mathcal{C}(\mathcal{F}, n)$. We also show that in several cases, the above mentioned extra additive term of $\log\log n + O(1)$ is necessary. In addition, using the notion of universal induced graphs, we show that the family of $n$-node graphs which are unions of disjoint circles enjoys an adjacency labeling scheme of size $\log n + O(1)$. This illustrates a non-trivial example showing that the above mentioned extra additive term is sometimes not necessary.

We then turn to investigate distance labeling schemes on the class of circles of size at most $n$. Using the notion of universal distance matrices we construct a distance labeling scheme for this family of size $1.5 \log n + O(1)$ and then show a lower bound of $\frac{4}{3}\log n - O(1)$ for the size of any such scheme.

Throughout, all additive constant terms are small, so no attempt was made to optimize them.

## 2 Preliminaries

Let $f$ be a function on pairs of vertices. An *$f$-labeling scheme* $\pi = \langle \mathcal{L}_\pi, \mathcal{D}_\pi \rangle$ for a graph family $\mathcal{F}$ is composed of the following components:

1. A *labeler* algorithm $\mathcal{L}_\pi$ that given a graph in $\mathcal{F}$, assigns labels to its vertices.
2. A polynomial time *decoder* algorithm $\mathcal{D}_\pi$ that given the labels $L(u)$ and $L(v)$ of two vertices $u$ and $v$ in some graph in $\mathcal{F}$, outputs $f(u, v)$.

The *size* of a labeling scheme $\pi = \langle \mathcal{L}_\pi, \mathcal{D}_\pi \rangle$ for a graph family $\mathcal{F}$ is the maximum number of bits in a label assigned by $\mathcal{L}_\pi$ to any vertex in any graph in $\mathcal{F}$.

We mainly consider the following functions on pairs of vertices $u$ and $v$ in a graph $G$.

The *adjacency* (respectively, *connectivity*) function: $f(u,v)$=1 if $u$ and $v$ are adjacent (resp., connected) in $G$ and 0 otherwise.
The *distance* function: $f(u,v) = d_G(u,v)$, the (unweighted) distance between $u$ and $v$ in $G$; we may sometimes omit the subscript $G$ when it is clear from the context.
The *flow* function: $f(u,v)$ is the maximum flow possible between $u$ and $v$ in a weighted graph $G$.

The *size* of a graph $G$, denoted $|G|$, is the number of vertices in it. Let $\mathcal{F}^{paths}(n)$ (respectively, $\mathcal{F}^{circles}(n)$) denote the family of paths (resp. circles) of size at most $n$. Given a family $\mathcal{F}(n)$ of connected graphs of size at most $n$, let $\mathcal{C}(\mathcal{F},n)$ denote the collection of graphs of size at most $n$, such that each graph in $\mathcal{C}(\mathcal{F},n)$ is composed of a disjoint union of some graphs in $\mathcal{F}(n)$.

A graph $G$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ is a *vertex induced subgraph* of a graph $\mathcal{U}$ with vertex set $\{u_1, u_2, \ldots, u_k\}$ if there exist indices $1 \le s_1, s_2, \ldots, s_n \le k$ such that for every $i,j \in \{1,2,\ldots,n\}$, $v_i$ and $v_j$ are neighbors in $G$ iff $u_{s_i}$ and $u_{s_j}$ are neighbors in $\mathcal{U}$. Given a graph family $\mathcal{F}$, a graph $\mathcal{U}$ is $\mathcal{F}$-induced universal if every graph in $\mathcal{F}$ is a vertex induced subgraph of $\mathcal{U}$.

**Proposition 1.** **[16]** *A graph family $\mathcal{F}$ has an adjacency labeling scheme with label size $g$ iff there exists an $\mathcal{F}$-induced universal graph with $2^g$ nodes.*

The *dimension* of a square matrix $M$, denoted $\dim(M)$, is the number of rows in it. An $n \times n$ square matrix $B = (b_{i,j})_{1 \le i,j \le n}$ is an *induced sub-matrix* of a $k \times k$ square matrix $A = (a_{i,j})_{1 \le i,j \le k}$ if there exists a sequence $(s_1, s_2, \ldots, s_n)$ of distinct indices $1 \le s_\ell \le k$ such that $b_{i,j} = a_{s_i,s_j}$ for every $i,j \in \{1,2,\ldots,n\}$. As defined in [10], given a graph family $\mathcal{F}$, an *$\mathcal{F}$-universal distance matrix* is a square matrix $M$ containing the distance matrix of every graph in $\mathcal{F}$ as an induced sub-matrix.

**Proposition 2.** **[10]** *If a graph family $\mathcal{F}$ enjoys a distance labeling scheme with label size $g$, then there exists an $\mathcal{F}$-universal distance matrix of dimension $2^{g+O(1)}$. Conversely, if there exists an $\mathcal{F}$-universal distance matrix of dimension $2^g$ then $\mathcal{F}$ enjoys a distance labeling scheme of size $g + O(1)$.*

Let $G$ be an $n$-node graph and let $u_1, u_2, \ldots, u_n$ denote its vertices. Given a function $f$ on pairs of vertices, the *$f$-matrix of $G$* is an $n$-dimensional square matrix $B$ such that $B_{i,j} = f(u_i, u_j)$ for every $i,j \in \{1,2,\ldots,n\}$. We first notice that the notion of a universal distance matrix can be extended into a *universal $f$ matrix* for any type of function $f$ on pairs of vertices. Formally, given a graph family $\mathcal{F}$, an *$\mathcal{F}$-universal $f$-matrix* is a square matrix $M$ containing the $f$-matrix of every graph in $\mathcal{F}$ as an induced sub-matrix of $M$. Going along the same steps as the proof of Proposition 2 in [10], we obtain the following proposition.

**Proposition 3.** *If a graph family $\mathcal{F}$ enjoys an $f$-labeling scheme with label size $g$, then there exists an $\mathcal{F}$-universal $f$-matrix of dimension $2^{g+O(1)}$. Conversely, if there exists an $\mathcal{F}$-universal $f$-matrix of dimension $2^g$ then $\mathcal{F}$ enjoys an $f$-labeling scheme of size $g + O(1)$.*

## 3 Transforming $f$-labeling schemes for connected graphs to non-connected graphs

### 3.1 The general transformation

Let $\mathcal{F}(n)$ be a family of connected graphs, each of size at most $n$. In this section we show that in many cases one can transform an $f$-labeling scheme for $\mathcal{F}(n)$ to an $f$-labeling scheme for $\mathcal{C}(\mathcal{F}, n)$ with a size increase of $\log\log n + O(1)$. We then show that this additive term is necessary in some cases but not always.

Let $f$ be a function on pairs of vertices, with the property that there exists some value $\sigma \in [0, \infty]$ such that $f(u, v) = \sigma$ for every two non-connected vertices $u$ and $v$. For example, $f$ can be the distance function (with $\sigma = \infty$), the flow function (with $\sigma = 0$) or the adjacency function (with $\sigma = 0$). First note that there exists a straightforward translation allowing us to transform a given $f$-labeling scheme $\pi = \langle \mathcal{L}, \mathcal{D} \rangle$ for a graph family $\mathcal{F}(n)$ of size $g(n)$ into an $f$-labeling scheme $\pi' = \langle \mathcal{L}', \mathcal{D}' \rangle$ for $\mathcal{C}(\mathcal{F}, n)$ of size $g(n) + \log n + O(1)$ as follows. Given a graph $H \in \mathcal{F}(n)$ and a vertex $v \in H$, let $L_H(v)$ be the label given to $v$ by the labeler algorithm $\mathcal{L}$ applied on $H$. Given a graph $G \in \mathcal{C}(\mathcal{F}, n)$, let $G_1, G_2, \ldots, G_k$ be its connected components (which belong to $\mathcal{F}(n)$). For $1 \le i \le k$, given a vertex $v \in G_i$, the label $\mathcal{L}'(v)$ to be assigned to $v$ by the labeler algorithm $\mathcal{L}'$ is composed of the concatenation of two sublabels, $M'_1(v)$ and $M'_2(v)$. The first sublabel, $M'_1(v)$, consists of precisely $\lceil \log n \rceil$ bits which are used to encode $i$ (padded by 0's to the left as necessary). The second sublabel is $M'_2(v) = L_{G_i}(v)$. Since the first sublabel consists of precisely $\lceil \log n \rceil$ bits, given a label $\mathcal{L}'(v)$ one can easily distinguish the two sublabels $\mathcal{L}'_1(v)$ and $\mathcal{L}'_2(v)$. Given two labels $\mathcal{L}'(v)$ and $\mathcal{L}'(u)$ of two vertices $v$ and $u$ in $G$, the decoder $\mathcal{D}'$ outputs $\mathcal{D}(\mathcal{L}'_2(v), \mathcal{L}'_2(u))$ if the two labels agree in their first sublabels, and $\sigma$ otherwise. Clearly $\pi'$ is an $f$-labeling scheme of size $g(n) + \log n + 1$ for $\mathcal{C}(\mathcal{F}, n)$.

**Lemma 1.** *Let $g(n)$ be an increasing function satisfying (1) $g(n) \ge \log n$ and (2) for every constant $c \ge 1$, $2^{g(c \cdot n)} \ge c \cdot 2^{g(n)}$*[4]*. If, for every $m \le n$, there exists an $f$-labeling scheme $\pi(m)$ of size $g(m)$ for $\mathcal{F}(m)$ then there exists an $f$-labeling scheme of size $g(n) + \log\log n + O(1)$ for $\mathcal{C}(\mathcal{F}, n)$.*

*Proof.* For every $1 \le i \le n$, let $m_i = \lfloor n/i \rfloor$ and let $M_i$ be the $\mathcal{F}(m_i)$-universal $f$-matrix obtained from the $f$-labeling scheme $\pi(m_i)$. Let $M$ be the matrix

$$\mathbf{M} = \begin{pmatrix} M_1 & \sigma & \sigma & \ldots \\ \sigma & M_2 & \sigma & \ldots \\ \sigma & \sigma & M_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where all entries except for the diagonal of $M_i$'s contain the value $\sigma$. Since the size of $\pi(m_i)$ is at most $g(n/i)$, we obtain that $\dim(M_i) \le 2^{g(n/i)+O(1)}$. By our

[4] These requirements are satisfied by all functions of the form $\alpha \log^{\beta} n$, where $\alpha, \beta \ge 1$.

assumption on $g(n)$, we get that $\dim(M_i) \leq 2^{g(n)}/i + O(1)$, hence $\dim(M) \leq O(n) + 2^{g(n)} \sum_{i=1}^{n} 1/i \leq O(n) + 2^{g(n)} \cdot \log n$. Since we assume $g(n) \geq \log n$, we obtain that $\dim(M) = O(2^{g(n)} \cdot \log n)$.
The lemma follows once we show that $M$ is an $\mathcal{C}(\mathcal{F}, n)$-universal distance matrix. Let $G = \bigsqcup_{i=1}^{k} G_i$ be a graph in $\mathcal{C}(\mathcal{F}, n)$ such that for every $1 \leq i < k$, $|G_i| \geq |G_{i+1}|$. If follows that for every $1 \leq i \leq k$, $|G_i| \leq m_i$. For each $1 \leq i \leq k$, using the $\mathcal{F}(m_i)$-universal $f$-matrix $M_i$, we map the vertices of $G_i$ to the corresponding indices of $M_i$ in $M$. ∎

### 3.2 Labeling schemes for path collections

The following easy to prove claim shows that the requirements from $g(n)$ in the previous lemma are necessary.

*Claim.* The size of a connectivity labeling scheme on $\mathcal{F}^{paths}(n)$ is $O(1)$ and the size of a connectivity labeling scheme on $\mathcal{C}(\mathcal{F}^{paths}, n)$ is $\Omega(\log n)$.

Note that one can easily show that the size of a distance labeling scheme for $\mathcal{F}^{paths}(n)$ is $\log n + \Theta(1)$. Therefore, the following lemma shows, in particular, that the extra additive term of $\log\log n + O(1)$ mentioned in Lemma 1 is sometimes necessary.

**Lemma 2.** *Any distance labeling schemes for $\mathcal{C}(\mathcal{F}^{paths}, n)$ must have size at least $\log n + \log\log n - O(1)$.*

*Proof.* It is enough to show that any $\mathcal{C}(\mathcal{F}^{paths}, n)$-universal distance matrix must have dimension $\Omega(n \log n)$. For simplicity of presentation, we assume that $n = m!$ for some $m$. The general case follows using similar arguments. Let $M$ be any $\mathcal{C}(\mathcal{F}^{paths}, n)$-universal distance matrix and let $k = \dim(M)$, i.e., $M$ is a square $k \times k$ matrix.

For every $1 \leq i \leq n$, let $G_i$ be the graph consisting of a disjoint union of $i$ paths of size $n/i$ each. We now define inductively $n$ sets of integers, $X_1, \ldots, X_n$, with the following properties.

1. $X_i \subset \{1, 2, \ldots, k\}$,
2. $X_i \subset X_{i+1}$,
3. $|X_n| = \Omega(n \log n)$,
4. for every $1 \leq i \leq n$, the set $X_i$ can be partitioned into $i$ disjoint subsets $Q_1, Q_2, \ldots, Q_i$, such that the following property is satisfied.
   **Partition property:** for every $1 \leq j \leq i$ and every two integers $x, y \in Q_j$, $M_{x,y} \neq \infty$.

The sets $X_1, \ldots, X_n$ are defined inductively as follows. Enumerate the vertices in $G_1$ from 1 to $n$, i.e., let the set of vertices in $G_1$ be $a_1, a_2, \ldots, a_n$. Let $X_1$ be a set of integers $\{s_1, s_2, \ldots, s_n\}$ satisfying that for every two vertices $a_h$ and $a_j$ in $G_1$, $f(a_h, a_j) = M_{s_h, s_j}$. The required properties trivially hold. Now assume that the sets $X_j$ are already defined for each $j \leq i$. Then $X_{i+1}$ is constructed as follows. Let $Q_1, Q_2, \ldots, Q_i$ be the subsets of $X_i$ satisfying the partition property and let $P_1, P_2, \ldots, P_{i+1}$ be the $n/(i+1)$-node paths of $G_{i+1}$. Let $a_1, a_2, \ldots, a_n$ denote the set of vertices in $G_{i+1}$ and let $(s_1, s_2, \ldots, s_n)$ be a sequence of integers such that for every $1 \leq a_h, a_j \leq n$, $f(a_h, a_j) = M_{s_h, s_j}$. For every $1 \leq j \leq i{+}1$, let $\varphi(P_j)$ be the set $\{s_k \mid a_k \in P_j\}$. By the partition property, for every $1 \leq h \leq i$, there exists at most one $1 \leq j \leq i+1$ such that $Q_h \cap \varphi(P_j) \neq \emptyset$. Therefore, there exists some $j$ such that $Q_h \cap \varphi(P_j) = \emptyset$ for every $1 \leq h \leq i$. Let $X_{i+1} = X_i \cup \varphi(P_j)$. Clearly, the partition property is satisfied for $X_{i+1}$. Moreover, $|X_{i+1}| = |X_i| + n/(i+1)$, and therefore $|X_n| \approx n(1 + \frac{1}{2} + \ldots + \frac{1}{n}) = \Omega(n \log n)$. Since $|X_n| \leq k$, the lemma follows. ∎

It can be shown that $\mathcal{C}(\mathcal{F}^{paths}, n)$ enjoys an adjacency labeling scheme of size $\log n + O(1)$. It follows that the extra additive term of $\log\log n + O(1)$ (Lemma 1) is not necessary in this case. In the following subsection we show another example for the fact that this additive term is not necessary, using the notion of universal induced graphs.

### 3.3 An adjacency labeling scheme for $\mathcal{C}(\mathcal{F}^{circles}, n)$

In this subsection we consider adjacency labeling schemes for $\mathcal{C}(\mathcal{F}^{circles}, n)$. Clearly, any adjacency labeling scheme for $\mathcal{F}^{circles}(n)$ must have size at least $\log n$. We now describe an adjacency labeling scheme for $\mathcal{C}(\mathcal{F}^{circles}, n)$ of size $\log n + O(1)$.

Let us first note that the following straightforward labeling scheme for the class $\mathcal{C}(\mathcal{F}^{circles}, n)$ uses labels of size $3 \log n$. Given a graph $G \in \mathcal{C}(\mathcal{F}^{circles}, n)$, let $C_1, C_2, \ldots$ be the circles of $G$. For each $i$, enumerate the vertices of $C_i$ clockwise and label each node $u \in C_i$ by $\mathcal{L}(u) = \langle i, |C_i|, n(u) \rangle$, where $n(u)$ is the number given to $u$ in the above mentioned enumeration. Given the labels $\mathcal{L}(u)$ and $\mathcal{L}(v)$ of two vertices in $G$, the decoder can easily identify whether $u$ and $v$ belong to the same circle or not. If $u$ and $v$ do not belong to the same circle, then the decoder outputs 0. Otherwise, the adjacency between $u$ and $v$ in their common circle $C_i$ can easily be determined using $|C_i|, n(u)$ and $n(v)$.

An *even* (respectively, *odd*) circle is a circle of even (resp., odd) size. Let $\mathcal{F}^{e-circles}$ be the graph family containing all $m$-node circles, where $m \geq 8$ is an even number. Let us first describe an $\mathcal{C}(\mathcal{F}^{e-circles}, n)$-universal graph. Let $\mathcal{U}^{e-circles}$ be a $3 \times n$-grid graph (see the bottom graph in Figure 2). Consider some graph $G \in \mathcal{C}(\mathcal{F}^{e-circles}, n)$ and let $C_1, C_2, \ldots$ be its collection of disjoint circles. Map each circle $C_i$ into $\mathcal{U}^{e-circles}$ leaving a gap of one column between any two consecutive mapped circles. An example of such a mapping is depicted in Figure 1.

It follows that $\mathcal{U}^{e-circles}$ is an $\mathcal{C}(\mathcal{F}^{e-circles}, n)$-universal graph of size $O(n)$. In the full paper we describe how to extend $\mathcal{U}^{e-circles}$ to obtain an $\mathcal{C}(\mathcal{F}^{circles}, n)$-universal graph of size $O(n)$. The following lemma follows.

**Lemma 3.** *There exists an adjacency labeling scheme of size* $\log n + O(1)$ *for* $\mathcal{C}(\mathcal{F}^{circles}, n)$.

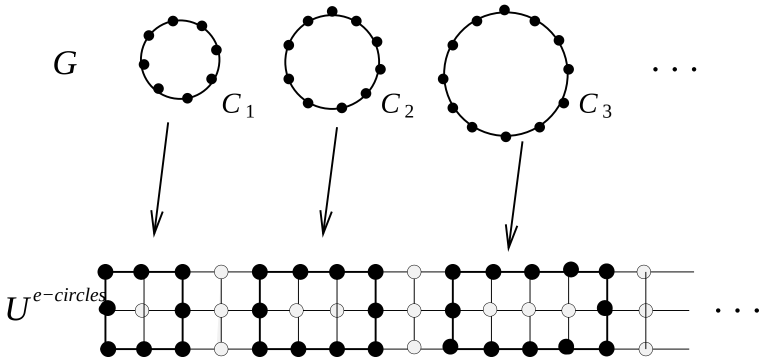


**Fig. 1.** A mapping of a graph $G \in \mathcal{C}(\mathcal{F}^{e-circles}, n)$ into the universal graph $\mathcal{U}^{e-circles}$.

# 4 A distance labeling scheme for $\mathcal{F}^{circles}(n)$

In this section we construct a distance labeling scheme for $\mathcal{F}^{circles}(n)$ of size $1.5 \log n + O(1)$ and establish a lower bound of $4/3 \log n - O(1)$ for the size of any such labeling scheme. Due to lack of space, this extended abstract contains only the lower bound proof.

## 4.1 A size lower bound on distance labeling schemes for $\mathcal{F}^{circles}(n)$

**Lemma 4.** *Any distance labeling scheme for $\mathcal{F}^{circles}(n)$ must have size at least $4/3 \log n - O(1)$.*

*Proof.* For simplicity of presentation, assume $n$ is divisible by 12. The general case follows using similar arguments. Let $\pi = \langle \mathcal{L}, \mathcal{D} \rangle$ be a distance labeling scheme for $\mathcal{F}^{circles}(n)$ and denote the set of labels assigned to the vertices of graphs in $\mathcal{F}^{circles}(n)$ by $X = \{\mathcal{L}(v) \mid v \in C,\ C \in \mathcal{F}^{circles}(n)\}$.

For $m = 1, 2, \ldots, n/12$, let $c_m = n/2 + 6m$ (note that $c_m$ is divisible by 6). For every $1 \leq m \leq n/12$, let $C_m$ be the circle with $c_m$ nodes. For any such circle $C_m$, let $I_m^1, I_m^2, \ldots, I_m^6$ be six vertex disjoint arcs of $C_m$, each of size $m/6$, ordered clockwise. Figure 2 shows this division of $C_{12}$ into 6 disjoint arcs. Define

$$\Psi_m = \{\langle \mathcal{L}(a), \mathcal{L}(b), \mathcal{L}(c) \rangle \mid a \in I_m^1, b \in I_m^3, c \in I_m^5\}.$$

It is easy to show that for every two vertices $v, u \in C_m$, $\mathcal{L}(v) \neq \mathcal{L}(u)$, therefore $\Psi_m$ contains $(c_m/6)^3 \geq (n/12)^3$ elements. Note that given any circle $C_m$, if $a \in I_m^1, b \in I_m^3$ and $c \in I_m^5$ then $d(a,b) + d(b,c) + d(c,a) = m$, so necessarily $\mathcal{D}(\mathcal{L}(a), \mathcal{L}(b)) + \mathcal{D}(\mathcal{L}(b), \mathcal{L}(c)) + \mathcal{D}(\mathcal{L}(c), \mathcal{L}(a)) = m$. We therefore obtain the following claim.

*Claim.* For every $1 \leq m < m' \leq n/12$, $\Psi_m \cap \Psi_{m'} = \emptyset$.

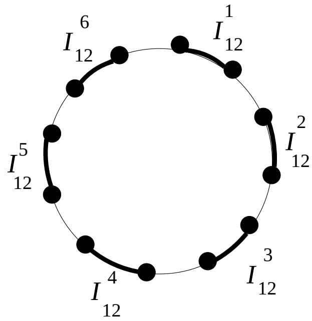


**Fig. 2.** The division of $C_{12}$ into 6 disjoint arcs $I^1_{12}, I^2_{12}, \ldots, I^6_{12}$.

Let $\Psi = \bigcup_{m=1}^{n/12} \Psi_m$. By the above claim, $\Psi$ contains $\Omega(n^4)$ distinct elements. Since $\Psi \subset X \times X \times X$, we obtain that $X$ contains $\Omega(n^{4/3})$ elements. Therefore, there exists a label in $X$ encoded using $\log(\Omega(n^{4/3})) = \frac{4}{3}\log n - O(1)$ bits. The lemma follows. ∎

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