

Garside monoids vs divisibility monoids

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Received January 2003

Divisibility monoids (*resp.* Garside monoids) are a natural algebraic generalization of Mazurkiewicz trace monoids (*resp.* spherical Artin monoids), namely monoids in which the distributivity of the underlying lattices (*resp.* the existence of common multiples) is kept as an hypothesis, but the relations between the generators are not supposed to necessarily be commutations (*resp.* be of Coxeter type). Here, we show that the quasi-center of these monoids can be studied and described similarly, and then we exhibit the intersection between the two classes of monoids. As a concrete illustration of the connection, we adapt a construction of an explicit finite transducer computing normal forms known for Garside monoids to the case of divisibility monoids.

1. Introduction

The purpose of this paper is to study some possible connections between the classes of Garside monoids and divisibility monoids, which are natural algebraic generalizations of monoids involved in two mathematical areas, namely braid theory and trace theory, respectively.

Garside monoids can be used as a powerful tool in the study of automatic structures for groups arising in topological or geometric contexts. Generalizing Mazurkiewicz trace monoids, divisibility monoids have been introduced as a mathematical model for the sequential behavior of some concurrent systems considered in several areas in computer science and in which two sequential transformations of the form ab and cd —with a, b, c, d viewed as atomic transitions—can give rise to the same effect.

Here we show that Garside monoids and divisibility monoids are both strangely similar (although they come from apparently unrelated mathematical theories) and genuinely different (even if they share some essential algebraic properties). Their similarity will be illustrated by the fact that their quasi-center—roughly speaking, some submonoid of the center—can be studied with close techniques and described through a same statement. Their particularity will be emphasized by the fact that the intersection between the two classes—that we exhibit by using the result about the quasi-center—is reduced to a somewhat confined subclass.

[†] The author thanks Prof. M. Droste and D. Kuske from the Institut für Algebra, TU Dresden, Germany, where he wrote this paper.

Main Theorem. (i) A divisibility monoid is a Garside monoid if and only if every pair of its irreducible elements admits common multiples.

(ii) A Garside monoid is a divisibility monoid if and only if the lattice of its simple elements is a hypercube.

The paper is organized as follows. In Section 1, we gather the needed basic properties of Garside monoids and divisibility monoids. In Section 2, we introduce a specific tool that we call *local delta* and which allows us to compute a minimal generating set for the quasi-center of every divisibility monoid (Propositions 3.12 and 3.14). We compare these results with those obtained for Garside monoids in (Picantin 2001b). In Section 3, we finally prove the main theorem of this paper (Theorem 4.1) and illustrate it.

2. Background from Garside and divisibility monoids

In this section, we list some basic properties of Garside monoids and divisibility monoids, and summarize results by Dehornoy & Paris about Garside monoids and by Droste & Kuske about divisibility monoids. For all the results quoted here, we refer the reader to (Dehornoy and Paris 1999; Picantin 2000; Dehornoy 2002) and to (Droste and Kuske 2001; Kuske 2001).

2.1. Divisors and multiples in a monoid

Assume that M is a monoid. We say that M is *conical* if 1 is the only invertible element in M . For a, b in M , we say that b is a left divisor of a —or that a is a right multiple of b —if $a = bd$ holds for some d in M . The set of the left divisors of b is denoted by $\downarrow(b)$. An element c is a right lower common multiple—or a right lcm—of a and b if it is a right multiple of both a and b , and every right common multiple of a and b is a right multiple of c . Right divisor, left multiple, and left lcm are defined symmetrically. For a, b in M , we say that b divides a —or that b is a divisor of a —if $a = cbd$ holds for some c, d in M .

If c, c' are two right lcm's of a and b , necessarily c is a left divisor of c' , and c' is a left divisor of c . If we assume M to be conical and cancellative, we have $c = c'$: the unique right lcm of a and b is then denoted by $a \vee b$. If $a \vee b$ exists, and M is left cancellative, there exists a unique element c satisfying $a \vee b = ac$: this element is denoted by $a \setminus b$. In particular, we obtain the identities :

$$a \vee b = a \cdot (a \setminus b) = b \cdot (b \setminus a).$$

Cancellativity and conicity imply that left and right divisibility are order relations. Now, the additional assumption that any two elements admitting a common multiple admit an lcm—that we will see is satisfied by both Garside monoids and divisibility monoids—allows to obtain the following algebraic properties for the operation \setminus .

Lemma 2.1. Assume that M is a cancellative conical monoid in which any two elements admitting a common right multiple admit a right lcm. Then the following identities hold in M :

$$(ab) \vee (ac) = a(b \vee c), \quad c \setminus (ab) = (c \setminus a)((a \setminus c) \setminus b), \quad (ab) \setminus c = b \setminus (a \setminus c),$$

$$(a \vee b) \setminus c = (a \setminus b) \setminus (a \setminus c) = (b \setminus a) \setminus (b \setminus c), \quad c \setminus (a \vee b) = (c \setminus a) \vee (c \setminus b).$$

For each identity, this means that both sides exist and are equal or that neither exists.

Notation 2.2. For some subsets A, B of elements, we denote by $A \setminus B$ the set of elements $a \setminus b$ provided that all of them exist.

2.2. The quasi-center of a monoid

The quasi-center of a monoid turns out to be a useful submonoid of its center.

Definition 2.3. Let M be a monoid. An *irreducible element* of M is defined to be a non trivial element a such that $a = bc$ implies either $b = 1$ or $c = 1$. The set of the irreducible elements in M can be written as $(M \setminus \{1\}) \setminus (M \setminus \{1\})^2$.

Definition 2.4. Assume that M is a monoid with set of irreducible elements Σ . The *quasi-center* of M is defined to be its submonoid $\{a \in M ; a\Sigma = \Sigma a\}$.

Although straightforward, the two following lemmas capture key properties of quasi-central elements. They will be frequently used in the remaining sections.

Lemma 2.5. Assume that M is a cancellative monoid. Then, for every element a in M and every quasi-central element b in M , the following are equivalent :

- (i) a divides b ;
- (ii) a divides b on the left;
- (iii) a divides b on the right.

Proof. By very definition, (ii) (*resp.* (iii)) implies (i). Now, assume (i). Then there exist elements c, d in M satisfying $b = cad$. Since b is quasi-central, we have $cb = bc'$ for some c' in M . We find $cb = bc' = cadc'$, hence, by left cancellation, $b = adc'$, which implies (ii). Symmetrically, (i) implies (iii). \square

Lemma 2.6. Assume that M is a cancellative monoid and a is a quasi-central element in M . Then, for any two elements b, c satisfying $a = bc$, b is quasi-central if and only if so is c .

Proof. Assume c to be quasi-central. Let x be an irreducible element in M . Since a is quasi-central, there exists an irreducible element x' in M satisfying $xa = ax'$, hence $xbc = bcx'$. Now, since c is quasi-central, there exists an irreducible element x'' in M satisfying $cx' = x''c$. We find $xbc = bx''c$, hence, by right cancellation, $xb = bx''$, which implies that b is quasi-central. Symmetrically, if b is quasi-central, so is c . \square

2.3. Main definitions and properties for Garside monoids

Definition 2.7. A monoid M is said to be *Garside*[‡] if M is conical and cancellative, every pair of elements in M admits a left lcm and a right lcm, and M admits a *Garside element*, defined to be an element whose left and right divisors coincide, are finite in number and generate M .

[‡] Garside monoids as defined above are called Garside monoids in (Charney *et al.* 2002; Dehornoy 2002; Picantin 2002; Picantin 2003a), but they were called either "small Gaussian" or "thin Gaussian" in previous papers (Dehornoy and Paris 1999; Picantin 2000; Picantin 2001a; Picantin 2001b), where a more restricted notion of Garside monoid was also considered.

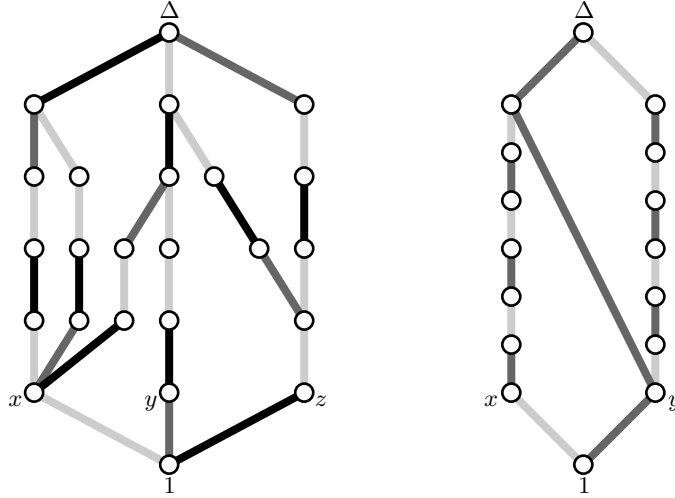


Fig. 1. The lattice of simple elements of the Garside monoids M_χ (left) and M_κ (right).

Example 2.8. All spherical Artin monoids are Garside monoids. Braid monoids of complex reflection groups (Broué *et al.* 1998; Picantin 2000), Garside’s hypercube monoids (Garside 1969; Picantin 2000), Birman-Ko-Lee monoids for spherical Artin groups (Birman *et al.* 1998; Picantin 2002) and monoids for torus link groups in (Picantin 2003a) are also Garside monoids.

The monoid M_χ with presentation

$$\langle x, y, z : xzxy = yzx^2, yzx^2z = zxyzx, zxyzx = xzxyz \rangle.$$

is a typical example of a Garside monoid, which has the distinguishing feature to be not antiautomorphic.

The monoid M_κ defined by the presentation $\langle x, y : xyxyxyx = yy \rangle$ is another example of a Garside monoid, which, as for it, admits no additive norm, *i.e.*, no norm ν satisfying $\nu(ab) = \nu(a) + \nu(b)$ (a *norm* for a monoid M is a mapping μ from M to \mathbb{N} satisfying $\mu(a) > 0$ for every $a \neq 1$ in M , and satisfying $\mu(ab) \geq \mu(a) + \mu(b)$ for every a, b in M).

Every element in a Garside monoid has finitely many left divisors, only then, for any two elements a, b , the left common divisors of a and b admit a right lcm, which is therefore the left gcd of a and b . This left gcd is denoted by $a \wedge b$.

Every Garside monoid admits a minimal Garside element, denoted in general by Δ . The set of the divisors of Δ —called the *simple elements*—endowed with the operations \vee and \wedge is a finite lattice.

Example 2.9. The lattices of simple elements of monoids M_χ and M_κ of Example 2.8 are displayed in Figure 1, using Hasse diagrams, where clear (*resp.* middle, dark) edges represent the irreducible element x (*resp.* y, z).

It is easy to check that every Garside element is a quasi-central element. Now, this is far from being sufficient to describe what the quasi-center of every Garside monoid looks like. The latter was done in (Picantin 2001b), where the following structural result was established.

Theorem 2.10. Every Garside monoid is an iterated crossed product of Garside monoids with an infinite cyclic quasi-center.

2.4. Main definitions and properties for divisibility monoids

Definition 2.11. A monoid M is called a *left divisibility monoid* or simply a *divisibility monoid* if M is cancellative and finitely generated by its irreducible elements, if any two elements admit a left gcd and if every element a dominates a finite[¶] distributive lattice $\downarrow(a)$.

Note that cancellativity and the lattice condition imply conicity. Like in the Garside case, the left gcd of two elements a, b will be denoted by $a \wedge b$. The *length* $|a|$ of an element a is defined to be the height of the lattice $\downarrow(a)$.

Example 2.12. Every finitely generated trace monoid is a divisibility monoid. The monoids $\langle x, y, z : xy = yz \rangle$, $\langle x, y, z : x^2 = yz \rangle$ and $\langle x, y, z : x^2 = yz, yx = z^2 \rangle$ are not trace but divisibility monoids. The monoid $\langle x, y, z : x^2 = yz, xy = z^2 \rangle$ is not a divisibility monoid (but a Garside monoid!); indeed, the lattice $\downarrow(x^3)$ is not distributive.

An easy but crucial fact about divisibility monoids is the following.

Lemma 2.13. Assume that M is a divisibility monoid. Then finitely many elements in M admitting at least a right common multiple admit a unique right lcm.

The following result states that there exists a decidable class of presentations that gives rise precisely to all divisibility monoids.

Theorem 2.14. Assume that M is a monoid finitely generated by the set Σ of its irreducible elements. Then M is a left divisibility monoid if and only if

- (i) $\downarrow(xyz)$ is a distributive lattice,
 - (ii) $xyz = xy'z'$ or $yzx = y'z'x$ implies $yz = y'z'$,
 - (iii) $xy = x'y'$, $xz = x'z'$ and $y \neq z$ imply $x = x'$,
- for any x, y, z, x', y', z' in Σ , and if
- (iv) we have $M \cong \Sigma^*/\sim$, where \sim is the congruence on Σ^* generated by the pairs (xy, zt) with x, y, z, t in Σ satisfying $xy = zt$.

3. The quasi-center of a divisibility monoid

This section deals only with divisibility monoids, and no knowledge of Garside monoids is needed—except possibly for the final remark. After defining the *local delta*, we use it as a fundamental tool in order to give a generating set for the quasi-center of every divisibility monoid and show that it is minimal. We establish then that the quasi-center of every divisibility monoid is a free Abelian submonoid.

[¶] This finiteness requirement is in fact not necessary since it follows from the other stipulations (see e.g. (Kuske 2001)).

3.1. A generating set for the quasi-center

We first introduce a partial version of what the author called *local delta* in (Picantin 2001b).

Definition 3.1. Assume that M is a divisibility monoid. Let a be an element in M . If the set $\{b \setminus a ; b \in M\}$ is well-defined[†] and admits a right lcm, the latter is called the *right local delta of a* or simply the *local delta of a* and is denoted by

$$\Delta_a = \bigvee \{b \setminus a ; b \in M\}.$$

Otherwise, we say that a does not admit a right local delta. Note that the equality $1 \setminus a = a$ implies a to be a left divisor of Δ_a , whenever it exists, and that, having $b \setminus 1 = 1$ for every b in M , we obtain $\Delta_1 = 1$.

Lemma 3.2. Assume that M is a divisibility monoid. Then an element a in M admits a local delta if and only if $a \vee b$ exists for every element b in M .

Proof. Assume that M is a divisibility monoid with Σ the set of its irreducible elements and $\Sigma_1 = \Sigma \cup \{1\}$. For every a in M , we define $\Upsilon_i(a)$ by $\Upsilon_0(a) = \{a\}$ and $\Upsilon_i(a) = \Sigma_1 \setminus \Upsilon_{i-1}(a)$ for $i > 0$ whenever both $\Upsilon_{i-1}(a)$ exists and $c \setminus b$ exists for every c in Σ_1 and every b in $\Upsilon_{i-1}(a)$.

Assume that a is an element such that $a \vee b$ exists for every b in M . From the distributivity of $\downarrow(a \vee b)$, we can deduce $|b \setminus a| \leq |a|$. The set $\{b \setminus a ; b \in M\}$ is then finite, and there exists a positive integer i_a satisfying

$$\{a\} = \Upsilon_0(a) \subsetneq \Upsilon_1(a) \subsetneq \dots \subsetneq \Upsilon_{i_a}(a) = \Upsilon_{i_a+1}(a) = \dots = \{b \setminus a ; b \in M\}.$$

Indeed, by Lemma 2.1, we have $M \setminus a = \Sigma_1^{i_a} \setminus a = \Sigma_1 \setminus (\Sigma_1^{i_a-1} \setminus a)$. Now, assume c, d belonging to $M \setminus a$ and h satisfying $d = h \setminus a$. By hypothesis, the element $(hc) \setminus a$ exists, hence, by Lemma 2.1, the element $c \setminus d$ —which is $c \setminus (h \setminus a)$ —exists, so does $c \vee d$. Therefore, the finite set $\{b \setminus a ; b \in M\}$ admits a right lcm, namely Δ_a .

The converse implication is straightforward. □

Remark 3.3. Following the previous proof, we can compute the right local delta of any element a by recursively computing the sets $\Upsilon_i(a)$ whenever they exist and then by computing the right lcm of $\Upsilon_{i_a}(a)$, which must exist by Lemma 3.2. See Examples 3.10 and 4.2 below.

We are going to prove that the quasi-center of any divisibility monoid is generated by the (possibly empty) set of the local delta of its irreducible elements (Proposition 3.9). The proof of this result relies on several preliminary statements.

Lemma 3.4. Assume that M is a divisibility monoid. Then, for every element a and every quasi-central element b in M , a dividing b implies Δ_a existing and dividing b .

Proof. By hypothesis and Lemma 2.5, there exists an element d in M satisfying $b = ad$. As b is quasi-central, for every c in M , there exists an element c' in M satisfying $cb = adc'$. By Lemma 2.13, for every c in M , $c \vee a$ —which exists and is $c(c \setminus a)$ —divides cb on the left, and, by

[†] that is, the element $b \setminus a$ exists—or equivalently, the element $a \vee b$ exists—for every b in M .

left cancellation, $c \setminus a$ exists and divides b on the left. Therefore, Δ_a exists (by Lemma 3.2) and divides b on the left. \square

A weaker but convenient version of Lemma 3.4 is the following :

Lemma 3.5. Assume that M is a divisibility monoid. Then every quasi-central element a in M satisfies $\Delta_a = a$.

The converse assertion is as follows :

Lemma 3.6. Assume that M is a divisibility monoid. Then every element a in M satisfying $\Delta_a = a$ is quasi-central.

Proof. Let x be an irreducible element of M . From $\bigvee(M \setminus a) = a$, we deduce that $x \setminus a$ is a left divisor of a . Therefore, $x(x \setminus a)$ is a left divisor of xa . Now, by definition, $x(x \setminus a)$ is $a(a \setminus x)$, so there exists d in M satisfying $xa = ad$. We obtain $|xa| = |ad|$, hence $|x| + |a| = |a| + |d|$. We deduce $|d| = |x| = 1$, thus d is an irreducible element of M . So, there exists a mapping f_a from the irreducible elements of M into themselves such that $xa = af_a(x)$ holds for every irreducible element x . By cancellativity, f_a is injective, hence surjective : a is quasi-central by definition. \square

Proposition 3.7. Assume that M is a divisibility monoid. Then, for every a in M , the element Δ_a is quasi-central, whenever it exists.

Proof. Let a be an element in M such that Δ_a exists. We claim that Δ_{Δ_a} exists and is Δ_a . By hypothesis, the set $\{b \setminus a ; b \in M\}$ is well-defined and admits a right lcm, namely Δ_a . As $\downarrow(\Delta_a)$ is finite (by definition of a divisibility monoid), so is the set $\{b \setminus a ; b \in M\}$. Therefore, M admits a finite subset T satisfying $M \setminus a = T \setminus a$. Let $T = \{c_1, \dots, c_r\}$. For every element b in M , the element $((c_1 b) \setminus a) \vee \dots \vee ((c_r b) \setminus a)$ exists and divides Δ_a on the left. By using Lemma 2.1, we find

$$((c_1 b) \setminus a) \vee \dots \vee ((c_r b) \setminus a) = b \setminus ((c_1 \setminus a) \vee \dots \vee (c_r \setminus a)) = b \setminus \Delta_a,$$

which implies that $b \setminus \Delta_a$ exists and divides Δ_a (on the left) for every b in M . We deduce that Δ_{Δ_a} exists (by Lemma 3.2) and divides Δ_a . Now, Δ_a dividing Δ_{Δ_a} , cancellativity and conicity imply $\Delta_{\Delta_a} = \Delta_a$. Therefore, by Lemma 3.6, Δ_a is quasi-central. \square

Corollary 3.8. Assume that M is a divisibility monoid. Then the partial application $a \mapsto \Delta_a$ is a surjection from M onto the quasi-center of M .

Proposition 3.9. Assume that M is a divisibility monoid with Σ the set of its irreducible elements. Then $\{\Delta_x ; x \in \Sigma\}$ is a generating set of the quasi-center of M .

Proof. Let b be a quasi-central element in M . We show using induction on the length $|b|$ of b that there exist an integer n and irreducible elements x_1, \dots, x_n satisfying $b = \Delta_{x_1} \cdots \Delta_{x_n}$. For $|b| = 0$, n is 0. Assume now $|b| > 0$. Then there exist an irreducible element x and an element b' in M satisfying $b = xb'$. By Lemma 3.4, Δ_x exists and we have $b = \Delta_x b''$ for some b'' in M with $|b''| < |b|$. By Proposition 3.7, the element Δ_x is quasi-central, hence, by Lemma 2.6, so is b'' . By induction hypothesis, there exist an integer m and irreducible elements y_1, \dots, y_m admitting local delta and satisfying $b'' = \Delta_{y_1} \cdots \Delta_{y_m}$. We obtain $b = \Delta_x \Delta_{y_1} \cdots \Delta_{y_m}$. \square

Example 3.10. Let us consider the following divisibility monoids $M_1 = \langle x, y, z : xy = yz, yx = zy \rangle$ and $M_2 = \langle x, y, z : xz = yx, yz = zx \rangle$. In both monoids, by Lemma 3.2, the irreducible elements x and z clearly do not admit right local delta. On the one hand, the quasi-center of M_1 is thus generated by the single element $\Delta_y = \bigvee\{y\} = y$. On the other hand, let us try to compute the local delta of y . According to Remark 3.3, we compute $\Upsilon_0(y) = \{y\}$, $\Upsilon_1(y) = \{1, x, y, z\} \setminus \{y\} = \{1, x, y, z\}$. Now, since $x \vee z$ does not exist, $\Upsilon_2(y)$ cannot exist either. Therefore, no irreducible element admits local delta : $\{\Delta_s ; s \in \{x, y, z\}\}$ is empty and the quasi-center of M_2 is then trivial.

3.2. Minimality of the generating set

We now prove that the generating set given in Proposition 3.9 is minimal.

Lemma 3.11. Assume that M is a divisibility monoid. Then any two irreducible elements x, y in M admitting local delta satisfy either $\Delta_x = \Delta_y$ or $\Delta_x \wedge \Delta_y = 1$.

Proof. We first prove that, for any two irreducible elements x, y admitting local delta and every b in M , $\Delta_x = b\Delta_y$ implies $b = 1$. According to Proposition 3.7, Δ_x and Δ_y are quasi-central, and, by Lemma 2.6, b is also quasi-central. Now, as $1 \setminus x = x$ holds, x divides Δ_x , and, by Lemma 2.5, we have $\Delta_x = dx$ for some d in M . We find

$$\Delta_x = dx = b\Delta_y,$$

and, by left cancellation, the element $d \setminus b$ —which exists by Lemma 2.13—divides x : since x is irreducible, $d \setminus b$ is either x or 1. Assume $d \setminus b = x$. Then, b being quasi-central, Lemma 3.5 implies $\Delta_b = b = \bigvee(M \setminus b)$, and, therefore, x divides b . By Lemma 3.4, Δ_x divides b , which, by cancellativity and conicity, implies $\Delta_y = 1$, a contradiction. We deduce $d \setminus b = 1$. We find then $\Delta_y = (b \setminus d)x$, and, by Lemmas 2.5 and 3.4, Δ_x divides Δ_y , which, by cancellativity and conicity, implies $b = 1$.

Finally, let x, y be irreducible elements in M admitting local delta. Assume $\Delta_x \wedge \Delta_y \neq 1$. Then there exists an irreducible element z in M dividing both Δ_x and Δ_y . By Lemma 3.4, Δ_z divides both Δ_x and Δ_y , which, by the result above, implies $\Delta_x = \Delta_z = \Delta_y$. \square

Proposition 3.12. Assume that M is a divisibility monoid with Σ the set of its irreducible elements. Then $\{\Delta_x ; x \in \Sigma\}$ is a minimal generating set of the quasi-center of M .

Proof. By Proposition 3.9, the set $\{\Delta_x ; x \in \Sigma\}$ generates the quasi-center of M . Let x be an irreducible element such that Δ_x exists, and a, b be quasi-central elements in M . It suffices to show that $\Delta_x = ab$ implies either $a = 1$ or $b = 1$. Assume $a \neq 1$. Then we have $a = ya'$ for some irreducible element y and some a' in M . As a is quasi-central, by Lemma 3.4, Δ_y exists and is a left divisor of a , and, therefore, Δ_y is a left divisor of Δ_x . We have $\Delta_y \neq 1$, hence, by Lemma 3.11, $\Delta_y = \Delta_x$. Cancellativity and conicity imply then $b = 1$. \square

3.3. A free Abelian submonoid

We conclude with the observation that the quasi-center of every divisibility monoid is a free Abelian submonoid (that is, a monoid isomorphic to a direct product of copies of the monoid \mathbb{N}).

Lemma 3.13. Assume that M is a divisibility monoid. Let a, b be elements in M admitting local delta. Then $a \vee b$ exists and admits a local delta, namely $\Delta_a \vee \Delta_b$.

Proof. Let a, b be elements in M admitting local delta. First, by Lemma 3.2, $a \vee b$ exists. Next, by Lemma 3.2 again, the element $b \vee c$ exists for every element c in M , so does $a \vee (b \vee c)$. Associativity of the operation \vee implies that $(a \vee b) \vee c$ exists for every element c in M . Therefore, by Lemma 3.2, $\Delta_{a \vee b}$ exists. Now, by Lemma 2.1, $(c \setminus a) \vee (c \setminus b) = c \setminus (a \vee b)$ holds for every c in M . We obtain then $\Delta_a \vee \Delta_b = \Delta_{a \vee b}$. \square

Proposition 3.14. Assume that M is a divisibility monoid. Let QZ be its quasi-center. Then QZ is a free Abelian submonoid of M , and the partial function $a \mapsto \Delta_a$ is a partial surjective semilattice homomorphism from (M, \vee) onto (QZ, \vee) .

Proof. Let Σ be the set of the irreducible elements in M . By Proposition 3.12, $\{\Delta_x ; x \in \Sigma\}$ is the minimal generating set of QZ . So, in order to prove that QZ is free Abelian, it suffices to show that, for any two x, y in Σ admitting local delta with $\Delta_x \neq \Delta_y$, the element $\Delta_x \setminus \Delta_y$ —which exists by Lemma 3.13—is Δ_y . Let x, y be irreducible elements admitting local delta with $\Delta_x \neq \Delta_y$. Then, by Lemma 3.11, we have $\Delta_x \setminus \Delta_y \neq 1$. Now, $\Delta_x \setminus \Delta_y$ divides Δ_y , and, by Lemma 3.13, the element $\Delta_x \setminus \Delta_y$ is quasi-central, which implies $\Delta_x \setminus \Delta_y = \Delta_y$ by Proposition 3.12. The second part of the assertion follows then from Lemma 3.13. \square

Both Propositions 3.12 and 3.14 hold for every Garside monoid. The corresponding proofs however are different, even though the current approach closely follows (Picantin 2001b). The point is that, in the Garside case, the function $a \mapsto \Delta_a$ is total and coincide with its left counterpart $a \mapsto \tilde{\Delta}_a$, but there need not exist an additive length (see for instance the monoid M_κ in Example 2.8), while, in the divisibility case, there exists an additive length but the function $a \mapsto \Delta_a$ is only partial and does not admit a left counterpart in general.

4. The intersection

We can finally state :

Theorem 4.1. (i) A divisibility monoid is a Garside monoid if and only if every pair of its irreducible elements admits common multiples.

(ii) A Garside monoid is a divisibility monoid if and only if the lattice of its simple elements is a hypercube.

Proof. (i) The condition is necessary by very definition of a Garside monoid. We have to show that it is also sufficient. Let M be a divisibility monoid with Σ the finite set of its irreducible elements. Assume that every pair of irreducible elements in M admits right multiples. Then, by Lemma 2.13, every pair of irreducible elements in M admits a unique right lcm, and, by Theorem 2.14 (iv) or simply by distributivity, such an lcm is of length 2. From this, a straightforward induction shows that any two elements a, b in M admit a unique right lcm. Therefore, by Lemma 3.2, Δ_a exists for every a in M . In particular, Δ_x exists for every irreducible element x of M . Finally, by Proposition 3.14, $\bigvee_{x \in \Sigma} \Delta_x$ is a minimal Garside element for M , thus M is a Garside monoid.

(ii) Let M be a Garside monoid with Σ the finite set of its irreducible elements and Δ its

minimal Garside element. Assume first that M is divisibility monoid. Then the finite lattice of its simple elements $\downarrow(\Delta)$ is distributive. By Proposition 3.14, we have $\Delta = \bigvee_{x \in \Sigma} \Delta_x$. Now, by Theorem 2.14 (iv), for every x in Σ , $M \setminus x$ is a subset of $\Sigma_1 = \Sigma \cup \{1\}$ including x , say $M \setminus x = \Sigma_x \cup \{x\}$, so we have $\Delta_x = \bigvee_{y \in \Sigma_x \cup \{x\} \subset \Sigma_1} y$, hence $\Delta = \bigvee_{x \in \Sigma} \Delta_x = \bigvee_{x \in \Sigma} x$. Therefore, the lattice $\downarrow(\Delta)$ is a finite distributive lattice, whose upper bound is the join of its atoms, so $\downarrow(\Delta)$ is a hypercube (see (Birkhoff 1967) or for instance (Stanley 1998, page 107)).

Conversely, assume that the lattice $\downarrow(\Delta)$ is a hypercube, that is, $\downarrow(\Delta)$ is isomorphic to $\mathbf{1}^{|\Sigma|}$ with $\mathbf{1}$ the rank 1 chain. According to (Picantin 2001a, Proposition 3.12), for every positive integer k , every element a dividing Δ^k admits a unique decomposition as $a = b_1 \vee \dots \vee b_k$ with b_i \vee -indecomposable for $1 \leq i \leq k$, that is, $b_i = b'_i \vee b''_i$ implying either $b_i = b'_i$ or $b_i = b''_i$. Then, for every positive integer k , $\downarrow(\Delta^k)$ is isomorphic to $\mathbf{k}^{|\Sigma|}$ with \mathbf{k} the rank k chain, which is a distributive lattice. Now, for every element a in M , the lattice $\downarrow(a)$ is a sublattice of a lattice $\downarrow(\Delta^k)$ for some positive integer k . Every sublattice of a distributive lattice being distributive, the lattice $\downarrow(a)$ is thus distributive for every element a in M . Therefore, all the requirements in the definition of a divisibility monoid are gathered. \square

Example 4.2. Up to isomorphism, there are 2 (resp. 5, 23) Garside divisibility monoids with rank 2 (resp. 3, 4). Those with rank 2 are $\mathbb{N}^2 = \langle x, y : xy = yx \rangle$ and $\mathbb{K} = \langle x, y : x^2 = y^2 \rangle^{\ddagger\ddagger}$. Figure 2 (to be compared with Figure 1) shows the lattice of simple elements of each of the 5 Garside divisibility monoids with rank 3. Note that, by using Theorem 2.10, we can recognize (from the left to the right) the monoids \mathbb{N}^3 , $\mathbb{N} \bowtie \mathbb{N}^2$, $\mathbb{N} \times \mathbb{K}$, $\mathbb{N} \bowtie \mathbb{K}$ and some indecomposable (as a crossed product) monoid. By using Proposition 3.14, we compute that their quasi-center is isomorphic to \mathbb{N}^3 , \mathbb{N}^2 , \mathbb{N}^2 , \mathbb{N}^2 and \mathbb{N} , respectively.

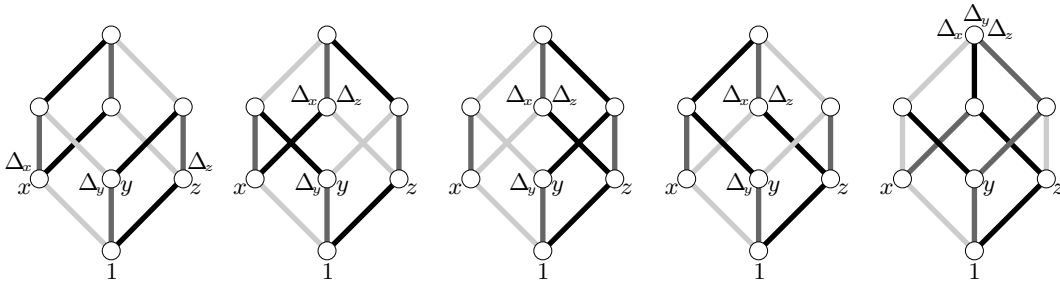


Fig. 2. The lattices of simple elements of the rank 3 Garside divisibility monoids.

In Remark 3.3 and Example 3.10, we have seen how local delta in a divisibility monoid are effectively computable. For instance, let us compute the quasi-center of the fifth rank 3 Garside divisibility monoid, namely $M_{3,5} = \langle x, y, z : x^2 = yz, y^2 = zx, z^2 = xy \rangle$. We have $\Upsilon_1(x) = \{1, x, y, z\} \setminus \{x\} = \{1, x, z\}$, $\Upsilon_2(x) = \{1, x, y, z\} \setminus \{1, x, z\} = \{1, x, y, z\} = \Sigma \cup \{1\} = \Upsilon_3(x) = M \setminus x$. Then x admits a local delta, namely $\Delta_x = \bigvee \{1, x, y, z\} = x^3$. Symmetrically, we have $\Delta_y = \Delta_z = x^3$. Therefore, the quasi-center of $M_{3,5}$ is quite isomorphic to \mathbb{N} .

$\ddagger\ddagger$ \mathbb{K} is for Klein Bottle.

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