POLYLOGARITHMIC NETWORK NAVIGABILITY USING COMPACT METRICS WITH SMALL STRETCH

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MOTIVATIONS

- Milgram Experiment in acquaintanceship networks (60’s):
  - “people are linked by short chains of acquaintances that can be discovered in a distributed manner”

- Kleinberg Model (2000):
  - Grids enhanced with long-range links distributed according to harmonic distributions $\approx 1/\text{dist}^d(x,y)$
A AUGMENTED GRAPH MODEL

- A pair \((G, \varphi)\) where
  - \(G\) is a graph
  - \(\varphi = \{ \varphi_u, u \in V(G) \}\) is a collection of probability distributions: \(\varphi_u(v) = \Pr(u \rightarrow v)\)
- One long-range link is added to every node \(u\), whose other extremity (long-range contact) \(v\) is chosen with probability \(\varphi_u(v)\)
GREEDY ROUTING

- \( H \in (G, \varphi) \)

- Source \( s \) and target \( t \)

- Current node \( x \) selects next node \( y \) as its neighbor closest to \( t \) in \( G \), i.e., according to \( \text{dist}_G \)

**Remarks:**

- Node \( x \) may select its long-range contact

- Selection according to \( \text{dist}_G \)
PERFORMANCES

Measure = maximum, taken over all pairs source-destination \((s,t)\), of the expected number of steps of greedy routing in \((G,\varphi)\) from \(s\) to \(t\).
POLYLOG GREEDY ROUTING: GRAPH CLASSES

- **Grids** [Kleinberg, STOC 2000]
- **Intersection systems** [Kleinberg, NIPS 2001]
- **Bounded ball growth** [Duchon, Hanusse, Lebhar, Schabanel, TCS 2006]
- **Bounded doubling dimension** [Slivkins, PODC 2005]
- **Trees** [Flammini, Moscardelli, Navarra, Perennes, DISC 2005]
- **Bounded treewidth** [F., ESA 2005]
- **Minor-excluding** [Abraham, Gavoille, PODC 07]
GREEDY ROUTING IN ARBITRARY GRAHS

- Uniform augmentation [Peleg 2006]: $O(\sqrt{n})$
- Best known upper bound
  - $O(n^{1/3})$ [F., Gavoille, Kosowski, Lebhar, Lotker, SPAA 2007]
- Best known lower bound
  - $\Omega(2^{\sqrt{\log n}})$ [F., Lebhar, Lotker, ESA 06]
- Bad news: polylogarithmic routing cannot be achieved using the distance metric of the base graph
Greedy routing in \((G, \varphi)\) according to \(\mu: V \times V \to \mathbb{R}^+\)

Greedy routing does not work for arbitrary function \(\mu\) (dead-end or ping-pong):

Example: discrete metric: \(\mu(x,y)=1 \iff x \neq y\)

Greedy routing converges if:

\(\mu\) is the distance metric in a spanner of \(G\).
Definition The navigability diameter, \( \text{nav}(G, \varphi, \mu) \), of \((G, \varphi, \mu)\) is the maximum, taken over all pairs source-destination, of the expected number of steps of greedy routing in \((G, \varphi)\) according to \(\mu\).

Remark For any \(G\), there exist \(\varphi\) and \(\mu\) such that \(\text{nav}(G, \varphi, \mu) \leq O(\log n)\).
THE NAVIGABILITY PROBLEM

Given any (connected) graph $G$, find

- an augmentation $\phi$
- a metric $\mu$

such that:

- $\text{nav}(G, \phi, \mu)$ is small ($\text{polylog}(n)$)
- $\mu$ has small stretch: $\max_{x,y} \mu(x,y) / \text{dist}_G(x,y)$
Our result: For any $n$-node connected graph $G$ with a positive edge cost function, there exist

- an augmenting distribution $\varphi$, and
- a semimetric $\mu$ with stretch $O(\log n)$,

such that $\text{nav}(G, \varphi, \mu) \leq O(\text{polylog}(n))$.

The semimetric $\mu$ can be encoded at every node using $O(\text{polylog}(n) \log(\Delta))$ bits.
**Theorem** For any $n$-node connected graph $G$ with a positive edge cost function, and any integer $k \geq 1$, there exist

- an augmenting distribution $\phi$, and
- a semimetric $\mu$ with stretch $2k-1$,

such that $\text{nav}(G,\phi,\mu) \leq O(k^2 n^{2/k} \log^2 n)$.

The semimetric $\mu$ can be encoded at every node using $O(k n^{1/k} \log n \log(k\Delta))$ bits, where $\Delta$ denotes the normalized diameter of $G$. 
**Proposition** There exist an edge-weighted graph $G$, a $(1+\varepsilon)$-spanner $S$ of $G$, and an augmentation $\varphi$ for $S$, such that:

- $\text{nav}(S,\varphi,\text{dists}) = O(\text{polylog } n)$
- $\text{nav}(G,\varphi,\text{dists}) = \Omega(n)$
PROOF OF THEOREM: TREE-COVER

- **Definition** A $(\sigma, \delta)$-tree-cover of $G$ is a collection $C$ of trees such that:
  - $\forall T \in C$, $T$ is a subgraph of $G$
  - $\forall x \in V(G)$, $\exists T \in C / x \in V(T)$
  - $\forall x, y \in V(G)$, $\exists T \in C / \text{dist}_T(x, y) \leq \sigma \cdot \text{dist}_G(x, y)$
  - $\forall x \in V(G)$, $|\{T \in C / x \in V(T)\}| \leq \delta$
**Theorem [Thorup, Zwick, J. ACM 2005]** Any graph has a \((2k-1, O(k n^{1/k}))\)-tree-cover, for all \(k \geq 1\).

**Lemma** If \(G\) has a \((\sigma, \delta)\)-tree-cover with \(\delta \leq n\), then there exist an augmenting distribution \(\varphi\), and a semimetric \(\mu\) with stretch \(\sigma\), such that \(\text{nav}(G, \varphi, \mu) \leq O(\delta^2 \log^2 n)\).

The semimetric \(\mu\) can be encoded at every node using \(O(\delta \log n \log(\sigma\Delta))\) bits.
THE SEMIMETRIC

- \( C_{u,v} \subset C \) is the set of trees containing \( u \) and \( v \).

- Setting: \( \mu(u,v) = \min_{T \in C_{u,v}} \text{dist}_T(u,v) \)

- Remark:
  - \( \mu \) has stretch \( \sigma \)
  - \( \mu \) does not satisfy the triangle inequality (it is a semimetric)
CENTROIDAL DECOMPOSITION
AUGMENTATION

1) Select a tree u.a.r. prob. $\geq \frac{1}{\delta}$
2) Select a centroid u.a.r. prob. $\geq \frac{1}{\log n}$
GREEDY ROUTING ANALYSIS

- Route from source $s$ to target $t$: $u_0, u_1, \ldots, u_r$
- $c_{j,k}$ denotes the centroid for $t$, of level $k$ in tree $j$ containing $t$.
- $\Phi(u) = \#\text{centroids } c_{j,k} \text{ closer to target than } u$
- $\Phi_{j,k}(u) = 1$ if $\mu(u, t) > \mu(c_{j,k}, t)$, and 0 otherwise

$$\Phi(u) = \sum_{j,k} \Phi_{j,k}(u)$$
ANALYSIS (2)

\[ \Phi(s) \leq \delta \log(n) \text{ and } \Phi(t) = 0 \]

\[ Z_i = \text{#steps to reduce } \Phi \text{ by at least 1 from } \Phi=i \text{ to } \Phi \leq i-1 \]

\[ \text{nav}(G, \varphi, \mu) \leq \sum_i E(Z_i) \]

Note: there are at most \( \delta \log(n) \) terms in the sum

Claim: \( E(Z_i) \leq \delta \log(n) \)
- $d_{j,k}$ denotes the centroid for current node $u$, of level $k$ in tree $j$ containing $u$.

- Let $j$ such that $\mu(u,t) = \text{dist}_{T_j}(u,t)$

Let $k$ be largest index such that $c_{j,k} = d_{j,k}$
If $u = c_{j,k} = d_{j,k}$ then $\Phi$ decreased by 1

Assume $u \neq c_{j,k} = d_{j,k}$

- $\mu(u,t) = \text{dist}_{T_j}(u,t)$
- $\text{dist}_{T_j}(u,t) = \text{dist}_{T_j}(u,d_{j,k}) + \text{dist}_{T_j}(d_{j,k},t)$
- $\mu(d_{j,k},t) \leq \text{dist}_{T_j}(d_{j,k},t) < \text{dist}_{T_j}(u,t)$

Thus if the long-range contact of $u$ is $c_{j,k} = d_{j,k}$ then $\Phi$ decreases by 1

This event occurs with probability $\geq 1/(\delta \log(n))$
Theorem [Gavoille, Peleg, Perennes, Raz, 2004] There exists a distance labeling scheme \((\lambda, \alpha)\) for trees with

- Labels \(\lambda(T,u)\) of size \(O(\log(n) \log(\Delta))\) bits
- Decoding \(\alpha\) of size \(O(1)\) and time \(O(1)\)

Labeling:

\[ L(G,u) = \left( (\beta_1, \lambda(T_{\beta_1},u)), (\beta_2, \lambda(T_{\beta_2},u)), \ldots, (\beta_\delta, \lambda(T_{\beta_\delta},u)) \right) \]
OPEN PROBLEMS

• Is it possible to design an augmenting distribution $\phi$, and a semimetric $\mu$ with constant stretch, such that $\text{nav}(G, \phi, \mu) \leq O(\text{polylog } n)$?

• Is it possible to design an augmenting distribution $\phi$, and a metric $\mu$ with polylog stretch, such that $\text{nav}(G, \phi, \mu) \leq O(\text{polylog } n)$ and $\mu$ can be encoded at every node using $O(\text{polylog } n)$ bits?
THANK YOU!