How To Explore a Fast Changing World

(On the Cover Time of Dynamic Graphs)

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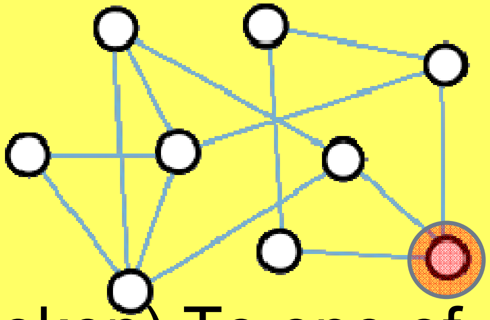
Joint work with Chen Avin & Michal Koucky (ICALP-08)

RW on Static Graphs

- The Simple Random Walk on Graph.

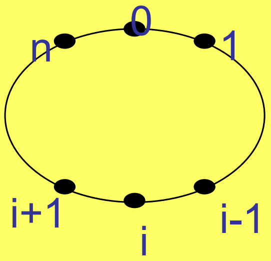


- Node sends messages (if it has the token) To one of his neighbors. The neighbor is chosen from the uniform distribution.
- Exhibits locality, simplicity, low-overhead, robustness
  - Becoming a popular approach for mobile devices

RW on Static Graphs

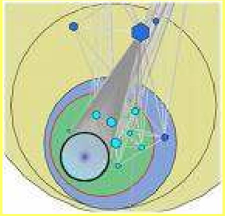
- Cover Time, hitting times are polynomial
- $C \leq 2m(n-1)$ [Aleliunas et al. 79]
- $(1-o(1))n\log n \leq C \leq (4/27+o(1))n^3$ [Feige95]

- For regular graphs C ≤ $O(n^2)$ [Kahn et al.89]

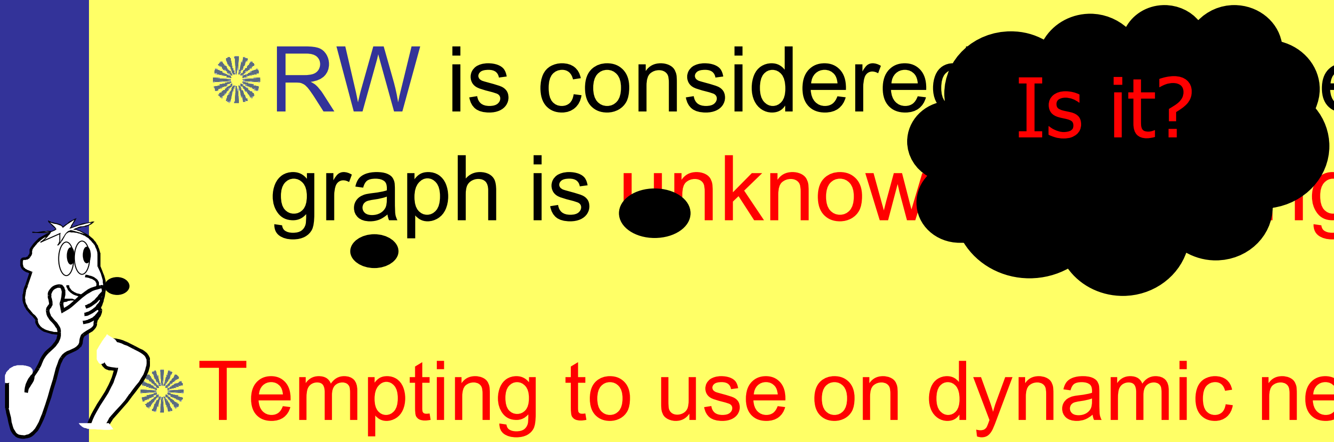


Application of RW

- Graph exploration: RW doesn't need to know the topology



  - RW is considere[illegible]en the graph is unknow[illegible]ging



- Tempting to use on dynamic networks

Talk Outline

- Dynamic graphs models and random walks on them
- Hitting time on dynamic graphs
- Connection between Dynamic graphs and Directed Graphs.
- “Nice” dynamic graphs
- Lazy Random Walk
- Discussion and open questions

Dynamic Model

- Evolving Graphs [Ferreira04] :

$$\mathcal{G} = G_1, G_2, G_3, \ldots \qquad \forall i,\ G_i = G(V, E_i)$$

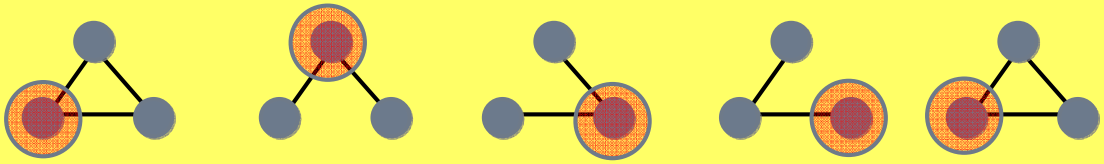
- Properties of evolving graphs:
  - $\mathcal{G}$ has property X if every G_i has property X.
- We concentrate on Explorable evolving graphs i.e. connected, self-loop.

Dynamic Model example

- Evolving Graphs:

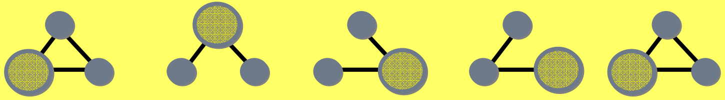
$$\mathcal{G} = G_1, G_2, G_3, \ldots \qquad \forall i,\ G_i = G(V, E_i)$$

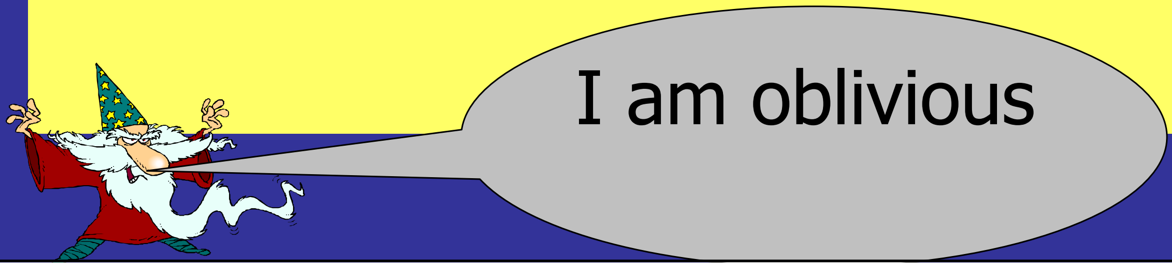
- Random walk on dynamic graph



G1 G2 G3 G4 G5 …

Random Walk on $\mathcal{G}$

- $\mathcal{G}$= 
  G_1 G_2 G_3 G_4 G_5 ...
- How do we measure the performance of Random walk on $\mathcal{G}$
  - Hitting time, cover time are defined as before.
- To prove that random walk is always "good" on dynamic graph, we consider worst case analyses.
  - A game between the walker and an oblivious adversary that controls the network dynamics.



Main Results

- **Question:** What will be the expected number of steps for a random walk on dynamic network to visit every node in the network (i.e., Cover Time).

- Answers in short:

  - Bad, very bad (compare to static network).

  - Can be fixed by the "Lazy Random Walk".

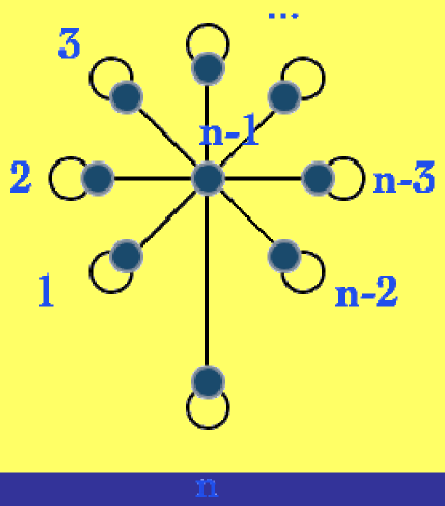
Lower Bound

- Lemma: The cover time of explorbale evolving graph is finite.

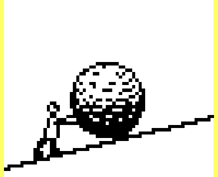
- Theorem: There exists an explorable evolving graph $\mathcal{G}$, such that the maximum hitting time of the simple random walk on $\mathcal{G}$ is $\Omega(2^n)$.

$$\mathcal{G} = G_1, G_2, G_3, \ldots$$

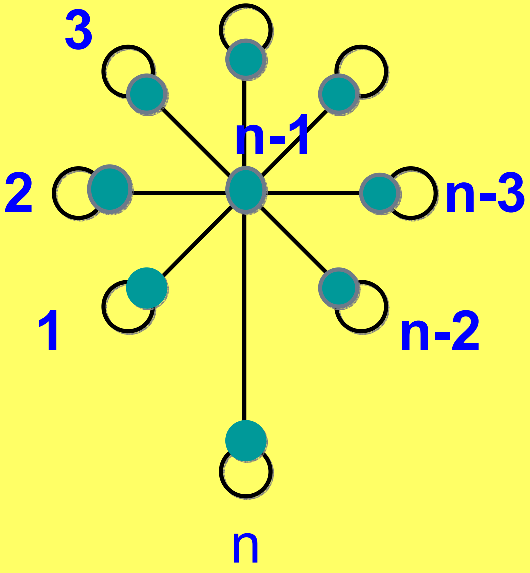
G_i =



“Sisyphus Wheel”

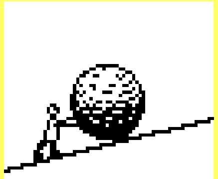


- The adversary has a simple (deterministic) strategy to increase h(1,n):



- The Cover Time of this dynamic graph is exponential!

“Sisyphus Wheel” Example



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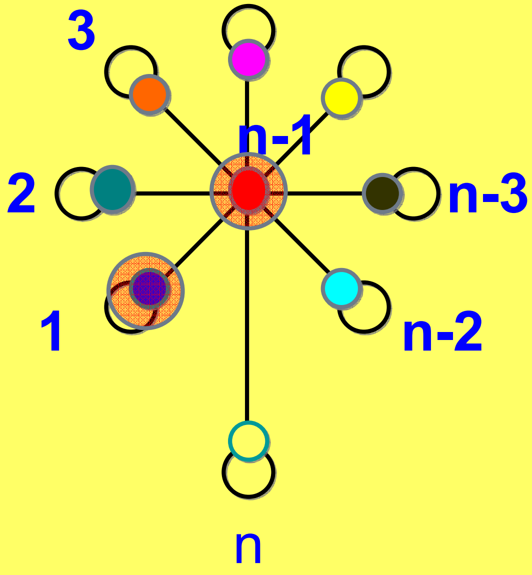

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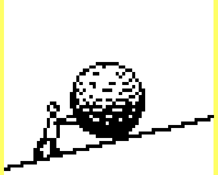

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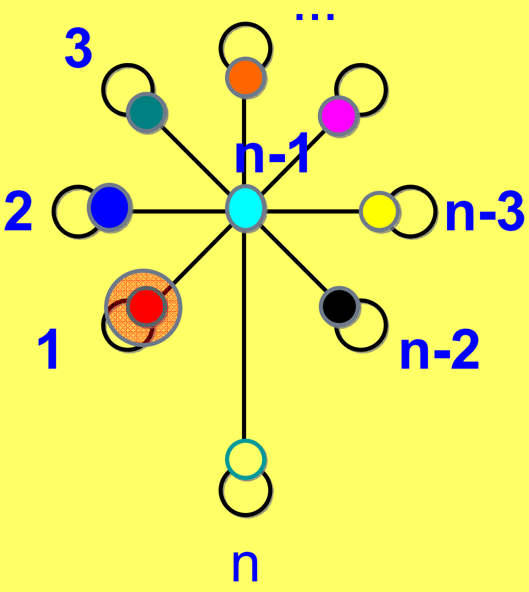

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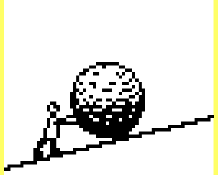

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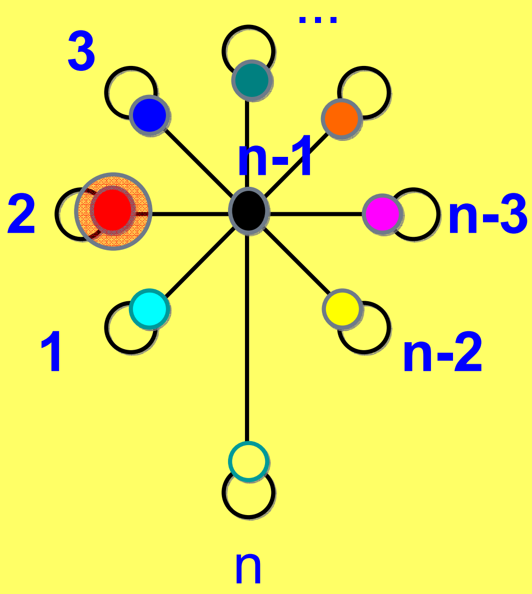
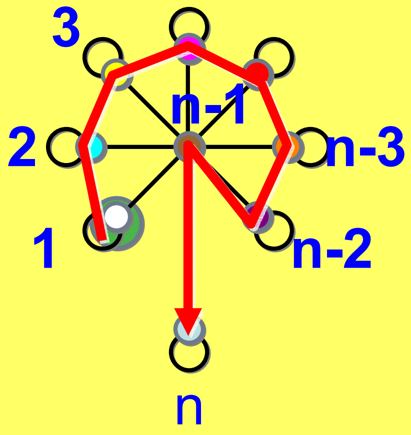


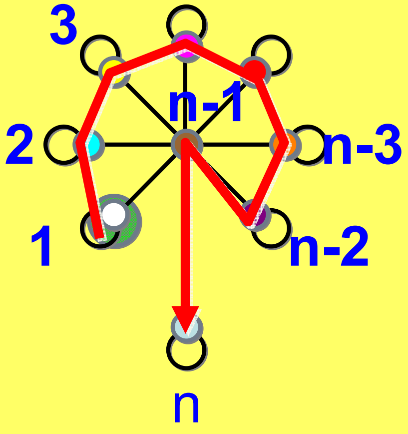
Exponential Hitting Time Proof idea for h(1,n)

- On condition that: we’ve never been at the center (node n-1)
  - after the adversary move, we have to reach node n-2 to get to the center. This requires n-2 consecutive self-loop steps, each with Pr. ½.



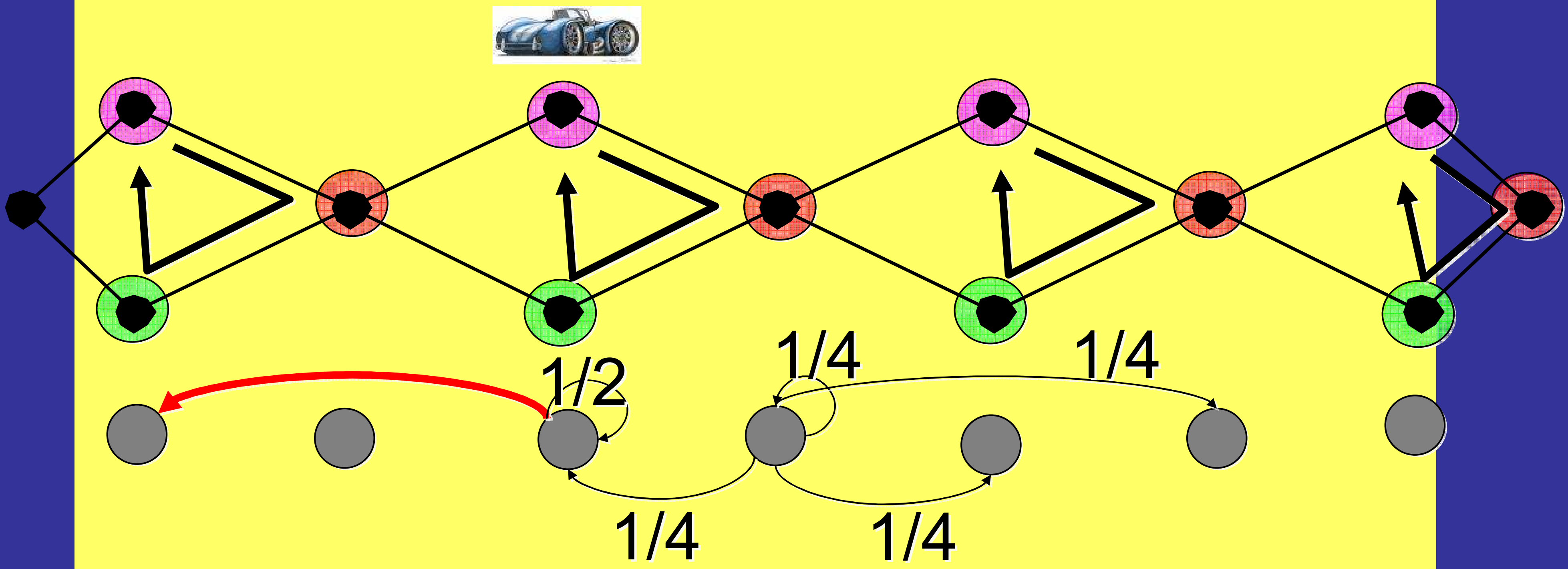
Exponential Hitting Time Proof idea for h(1,n)

- Note: All graphs in G are isomorphic, rapidly mixing with cover time of O(n log n).
  - Common tools from static graphs (e.g., spectral analysis) cannot be applied naively.



The connection between directed and Dynamic Graphs

- We assume that there are no self loops



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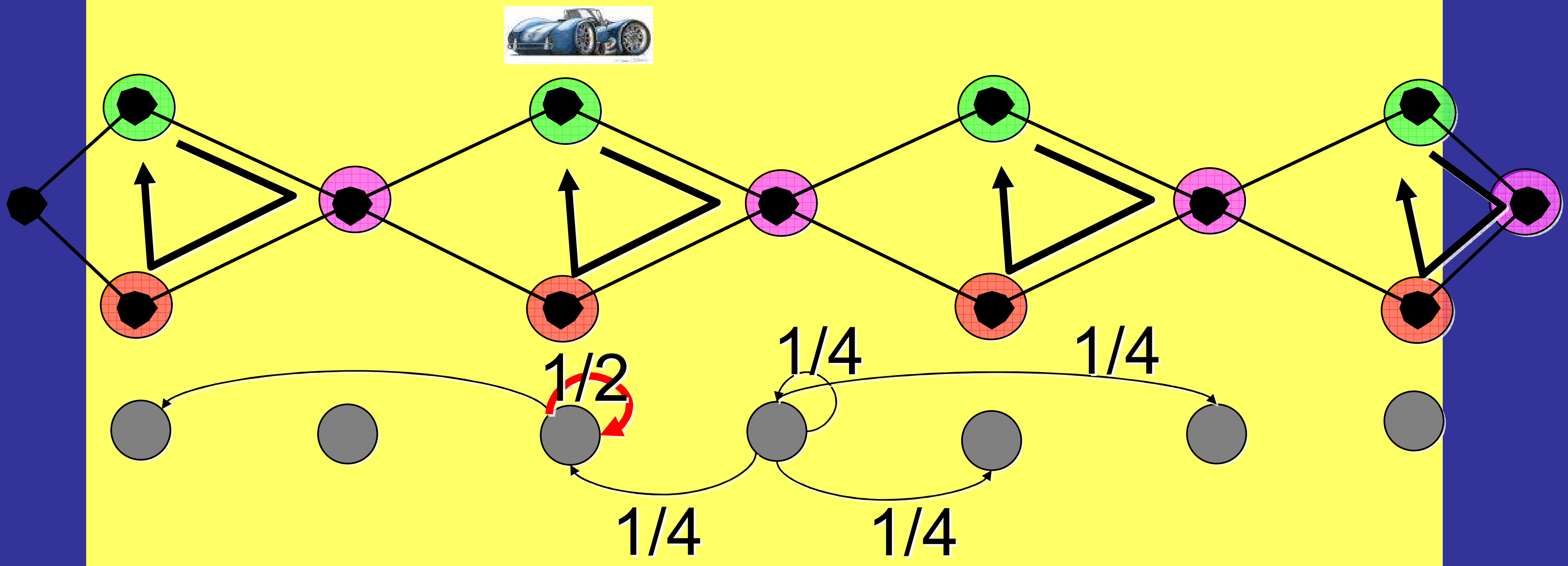

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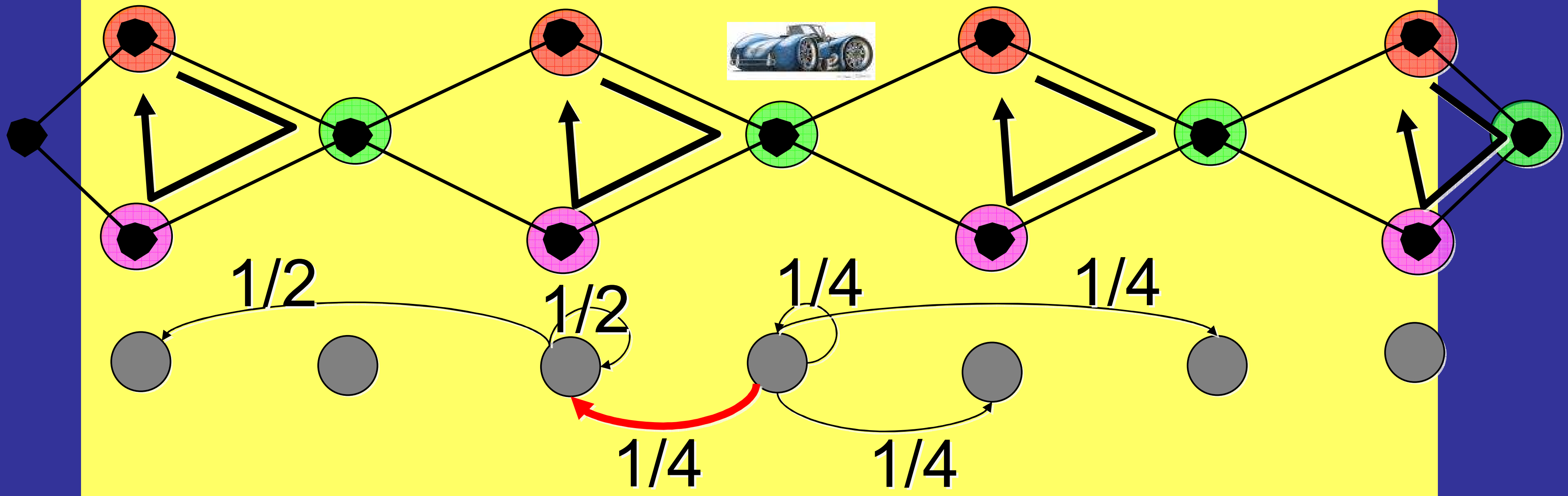

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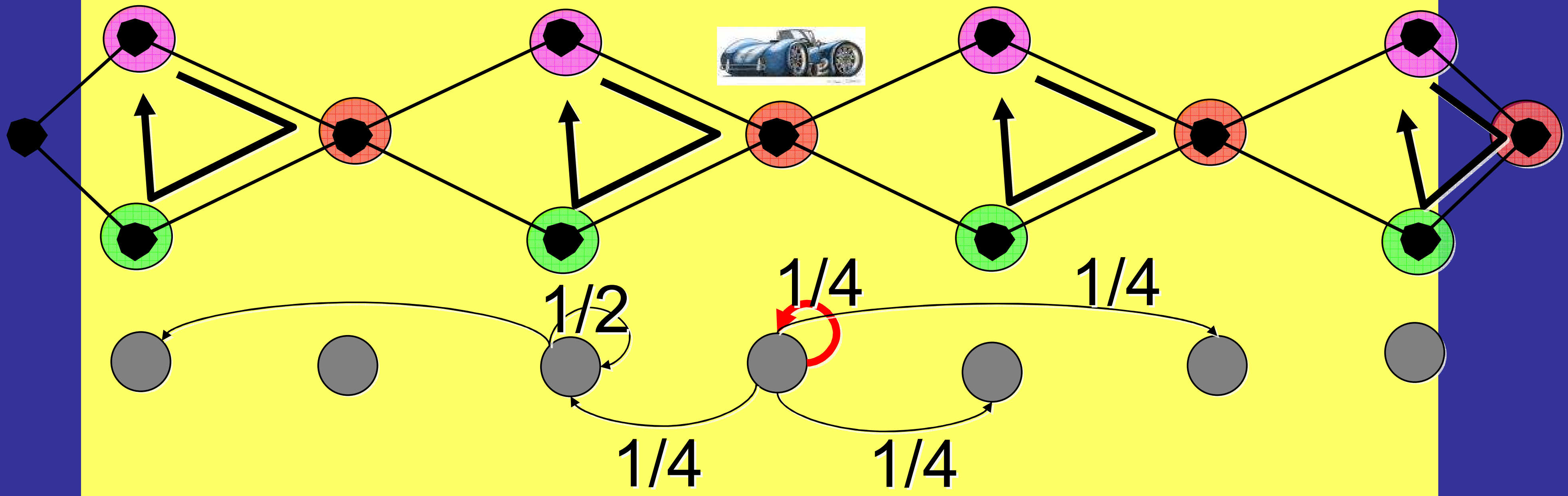

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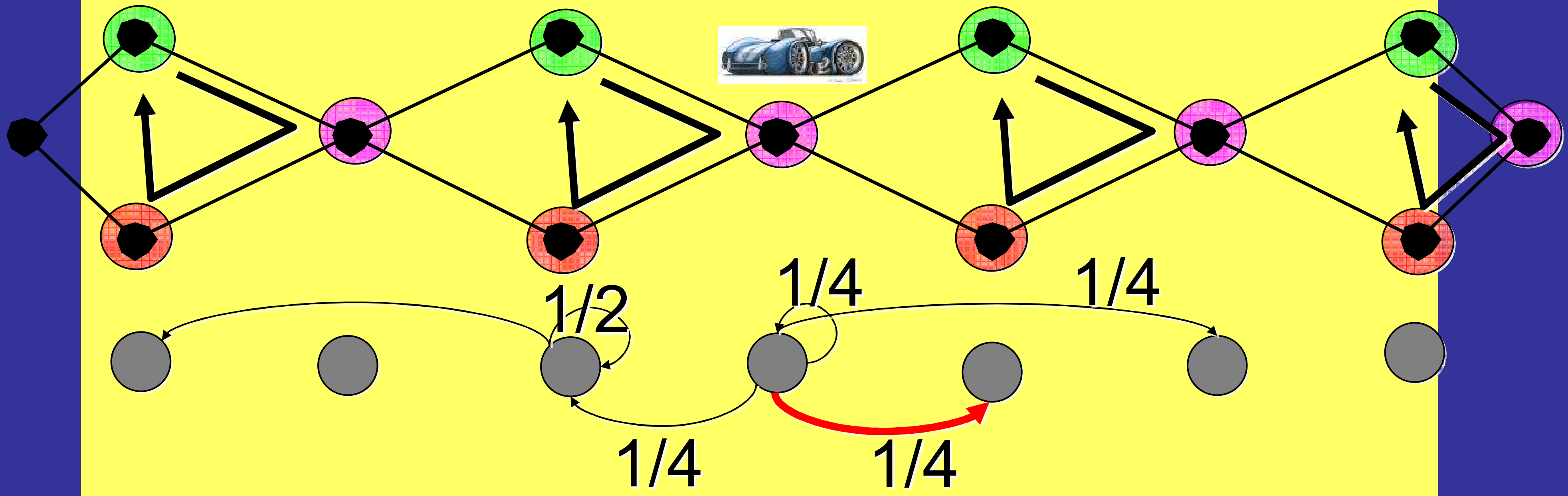
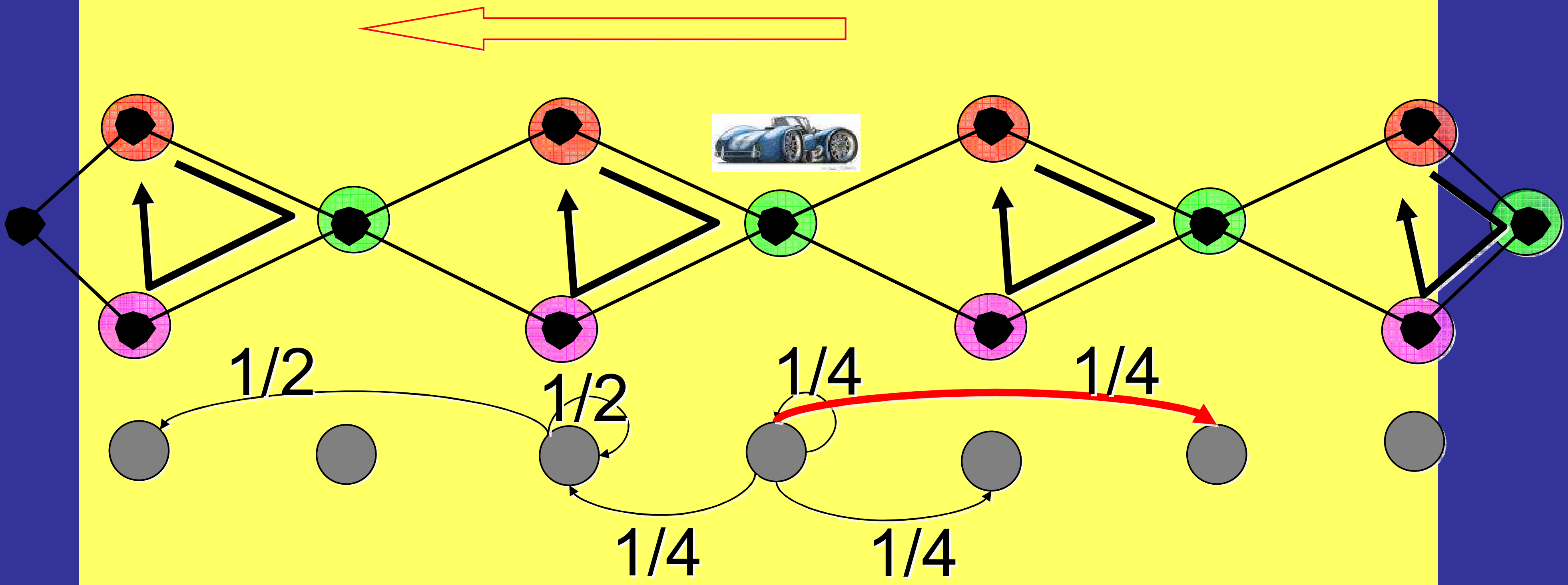


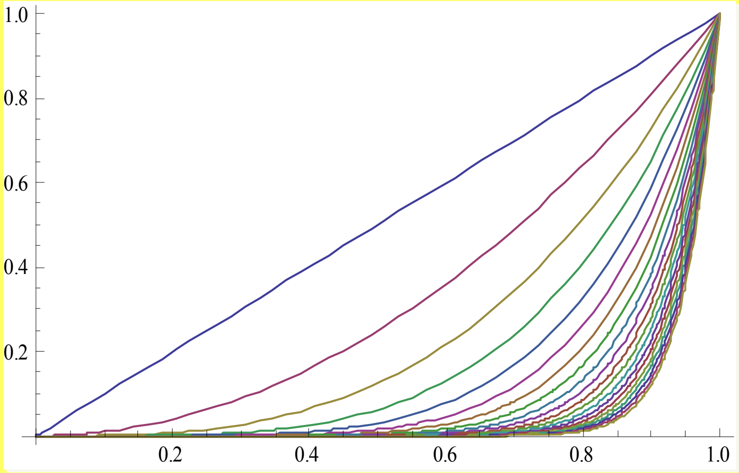
The connection between directed and Dynamic Graphs

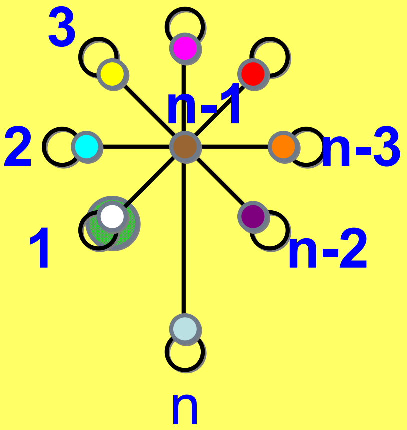
1/2-1/4=1/4



Regular Dynamic Graphs

- Theorem: For any d-regular connected non-bipartite evolving graph $\mathcal{G}$ the cover time of the simple random walk on $\mathcal{G}$ is $O(d^2n^3\ln^2n)$.





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- **Proof idea**:
  - There is a well define stationary distribution for $\mathcal{G}$, the uniform distribution $\pi = \frac{1}{n}$.
  - Can extend known bounds on the mixing time (the time to be anywhere) of regular grpahs.
  - Use coupon collection “type” argument.

Proof Outline

- **Let G be undirected, connected, non-bipartite, d-regular graph**. Let A be the transition matrix of the simple random walk on G. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. It is known that [Lovasz93]:

$$\lambda_1 = 1 \text{ and } \forall i \geq 2,\ \ \lambda_i \leq 1 - \frac{1}{dn^2} \text{ and } \forall i \geq 2,\ \ \lambda_i^2 \leq 1 - \frac{1}{d^2n^2}$$

- Lemma: for any probability distribution p on the vertices of G

$$\left\| pA - \frac{\mathbb{I}}{n} \right\|_2^2 \leq \left(1 - \frac{1}{d^2n^2}\right) \left\| p - \frac{\mathbb{I}}{n} \right\|_2^2$$

Proof (cont.)

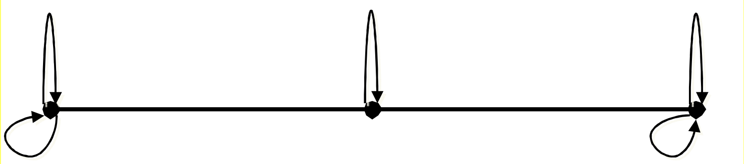
- Corollary: For connected, non-bipartite, d-regular evolving G = G_1, G_2, G_3, ...G_t, ...

$$\left\| p_t A - \frac{\mathbb{I}}{n} \right\|_2^2 \leq \left(1 - \frac{1}{d^2 n^2}\right)^t \left\| p_0 - \frac{\mathbb{I}}{n} \right\|_2^2$$

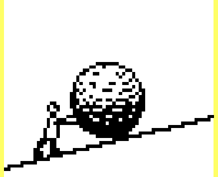
- If the walk is at u, after $t=4d^2n^2(\ln n)$ steps it can be at any node v with probability $> \frac{1}{2n}$. $(p_t(v) > \frac{1}{2n})$
- Now, to collect n coupons (nodes) we need O(n ln n) such rounds.

The Lazy Random Walk

- Lazy random walk: At each step of the walk pick a vertex v from V(G) uniformly at random and if there is an edge from the current vertex to the vertex v then move to v otherwise stay at the current vertex [JS83].



“Sisyphus Wheel” Example LRW



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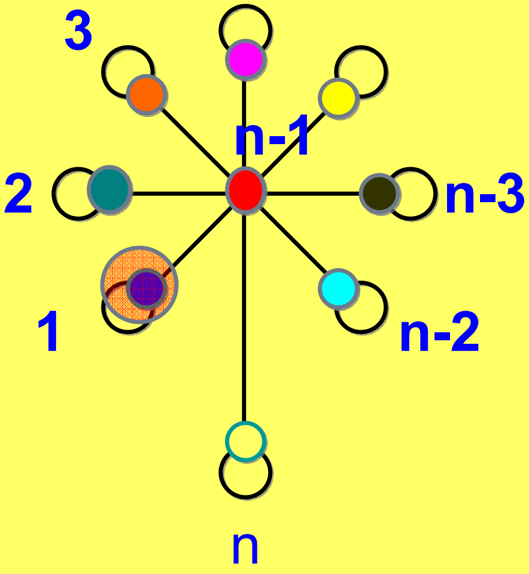

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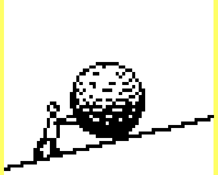

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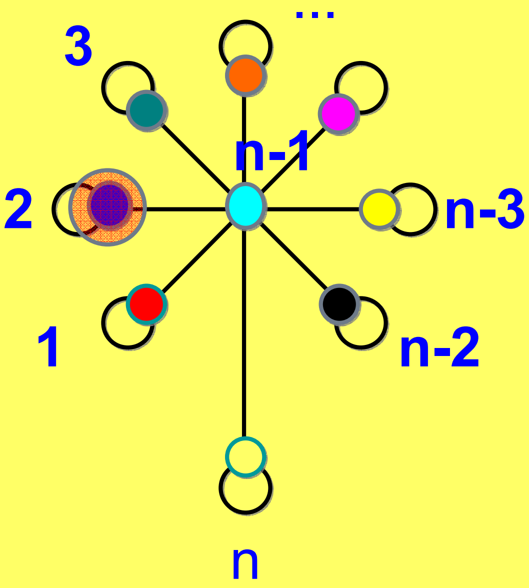

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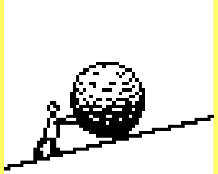

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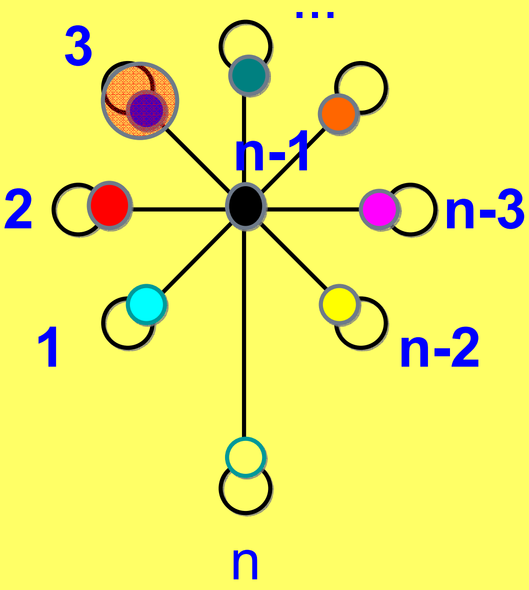


The Lazy Random Walk

- Theorem: For any connected evolving graph $\mathcal{G}$ the cover time of the lazy random walk on $\mathcal{G}$ is $O(n^5 \ln^2 n)$.
- ?? Slower is faster ?? :-)

Summary

- We showed that the cover time of the simple random walk on dynamic graphs is significantly different from the case of static graphs: exponential vs. polynomial.
- Dynamic graphs ==> simulating directed graphs
- The cover time is bounded to be polynomial by the use of lazy random walk.
- Gives some theoretical justification for the use of random-walks-techniques in dynamic networks, but careful attention is required.

Extensions

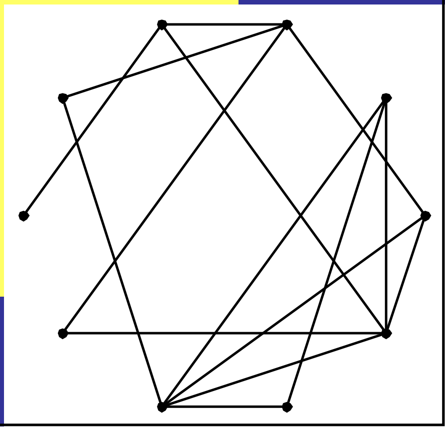
- Super-polynomial hitting time even in "slow"
- Unknown starting positions.
- Unknown max degree.
- Unknown network size.

Markovian evolving graph

- Dynamic Evolving Graphs $\mathcal{G}$ =(***G***,D)
- ***G*** is set of graphs, D is a (random) process
- Markov Evolving Graphs M=(***G***,*P*)

$\Pr[G_{t+1}=g|G_t=g_t,G_{t-1}=g_{t-1},\ldots,\ G_1=g_1]=\Pr[G_{t+1}=g|G_t=g_t]$

- Evolving graph where p is i.i.d, are Bernoulli.



Bernoulli Dynamic Graphs

- Theorem: For any explorable Bernoulli evolving graph, B=(***G***,*p*), the cover time of the simple random walk on B is O($n^3 \log n$) and the maximum hitting time is O(n^3).

- Proof idea: simple random walk on B is identical to (non-simple) random walk on a static graph *U*.

$$U = \bigcup_{g \in G} g$$

- Use standard techniques to bound the cover timeon *U*, i.e, resistance and Matthews' bound.

Open Questions

- Better, tighter bounds.
- best and worst dynamic graphs for simple rw.
- Tight bound for the lazy random walk.
- Limited adversary power (e.g., one change at a
- time).
- Directed dynamic graphs.

Thank You!