# The Dynamics of Probabilistic Population Protocols*

**Paul G. Spirakis**
**Coauthored with: I. Chatzigiannakis**

**DYNAMO 2008**

**(also in DISC 2008)**

* Research Academic Computer Technology Institute (RACTI)

September 2008

# Population Protocols

[Angluin, Aspnes, Diamandi, Fischer, Peralta, 04]

**Finite set of States** $Q=\{q_1, \ldots, q_k\}$

n agents (population)

$n_i$ = # agents in state $q_i$

Let $x_i = \dfrac{n_i}{n}$

**Configuration**, C: A map from the population to the states

**Population state at time t**

$x(t) = (x_1(t), \ldots, x_k(t))$

## Transition function

$\_ : QxQ \rightarrow QxQ$

**Probabilistic Population Protocols** (PPP), [Angluin, et al]

(all pairs interactions) (Fairness)

We generate $C_{t+1}$ from $C_t$ by (1) drawing an ordered pair (i, j) of agents **independently** and **uniformly**.

(2) Applying _ to $(C_k(i), C_k(j))$

(3) Updating the states of i,j, accordingly,

to form $C_{k+1}$

**Example** (Rules of _)

$$(q_1, q_2) \rightarrow (q_3, q_2)$$
$$(q_3, q_1) \rightarrow (q_1, q_2)$$
$$(q_2, q_3) \rightarrow (q_2, q_1)$$

(else no change)

How can we study systematically the (eventual) **stability** of such a system?

**Our proposal**: Use nonlinear dynamics of continuous time $\dot{x}_i = f_i(x) \quad i = 1 \ldots$_

Example:

$(q_1,q_2) \rightarrow (q_3, q_2)$

$(q_3, q_1) \rightarrow (q_1, q_2)$

$(q_2, q_3) \rightarrow (q_2, q_1)$

By inspection then

$$\dot{x}_1 = x_1x_3 + x_2x_3 - x_1(x_2 + x_3)$$

$$\dot{x}_2 = x_1x_3 + x_1x_2 + x_2x_3 - x_2(x_1 + x_3)$$

$$\dot{x}_3 = x_1x_2 - x_3(x_1 + x_2)$$

**Output rate** of a state i

$$\_{}_i(x) = \_x_m(t)$$

where $(q_i, q_m) \rightarrow (q_r, q\_)$

or $(q_m, q_i) \rightarrow (q_r, q\_)$

## Compare with the general Lotka-Volterra equation for 3 species

$$\dot{x}_i = x_i\left(r_i + \sum_{j=1}^{3} x_j a_{ij}\right)$$

$i = 1,2,3$

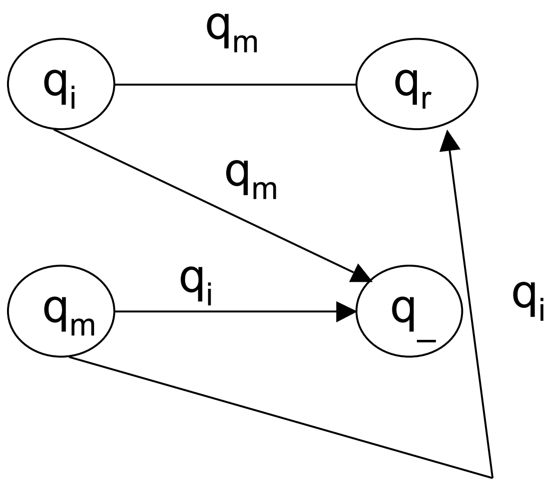
Here $r_i = -\_{i}$ and by setting $a_{ij}$ equal to 1 or 0 we get a special case of the dynamics of probabilistic population protocols, in which $q_i$ must be present in the left hand of a rule when it is present on the right.

- When are they equivalent?

## Note that we can represent

$$(q_i, q_m) \rightarrow (q_r, q\_)$$

as the following graph
(forgetting orders in the pair)




This produces a digraph G(_)

## Remark

Sources (and sinks) in G(_) are **transient** states in the sense that their population always decreases (increases) and then we can omit them from the differential equations approach, as far as eventual stability is concerned.

# I. When can we form such a system of differential equations?

[N. Wormald, 06] : In many cases

Note $n_i(t)$ are random variables

**(i)** $\exists$ Constant K : $|n_i\,(t+1) - n_i(t)| <$ _

(they don't change too quickly)

**(ii)** E ($n_i\,(t+1) - n_i(t)$ / history)

$$= f_i\left(\frac{t}{n}, x_1, \ldots, x_k\right) + 0(1)$$

(we know the **expected** rate of change)

**(iii)** $f_i$ is continuous and satisfies a Lipchitz condition.

Notice that the study of the rules of _, with (nonlinear) differential equations, cannot not be always applied.

For example, a probabilistic population protocol whose outcome depends on e.g. $n_1(t)$ being odd/even is impossible to be studied in this way.

Let $Z_i(t) = \frac{n_i(t)}{n(t)}$, a stochastic process.

**(the "real" process)**

**Definition**: The nonlinear dynamic system of equations _(_) corresponding to the rules of _ is **faithful** iff its solution $x_i(t)$ satisfies.

(1) $Z_i(0) = x_i(0)$

(2) $\forall \in >0$ and any integer T>0 there is a positive constant $N_o$ so that if $n>N_o$ then

Prob $\{|Z_i(t) - x_i(t)| \geq \in\} < \in$,
for any t : $0 \leq t \leq T$ at which $x_i(t)$ is defined.

## Remarks

**(a)** What are the faithful probabilistic population protocols?

(open)

**(b)** The faithfulness definition does not only speak about $t \to \infty$ but requires the sample paths of the PPP model to behave “according to their expected behaviour” in the long run.

**II.** Assume that a PPP is faithful. What conclusions do we get about its asymptotic stability?

How can we easily design (and understand) the behaviour of faithful PPP?

## II.1. A proposal for “globally” specifying a faithful PPP protocol.

The Switching Probabilistic Protocols

**idea**

**(A)** Specify when each agent in $q_i$ wants to **review** her state.

**(B)** If so, specify rates (probabilities) to go from $q_i$ to $q_j$.

## Switching Probabilistic Protocols (SPP)

A “simpler” dynamic system (global configurations)

**(a)** Each agent of state $q_i(t)$ **reviews** her state at rate _$_i(t)$

**(b) Conditional Rates of switch**

Each agent, in state $q_i(t)$, **switches to** state $q_j$ at a rate $p_{ij}(x(t))$

If all Review / Switch rates are **Poisson,** and independent, then when $n \rightarrow \infty$

$$\dot{x}_i = \sum_{j \in Q} x_j p_{ji}(x) \ddot{e}_j(x) - x_i \ddot{e}_i(x)$$

i =1…._

(no Wormald needed here)

## Theorem

The differential equations of SPP include those of PPP as a special case.

(Almost by inspection)

In our example choose

$$\_{1} = x_2 + x_3$$
$$\_{2} = x_1 + x_3$$
$$\_{3} = x_1 + x_2$$

$$p_{21} = \frac{x_3}{x_1 + x_3} \qquad p_{11} = \frac{x_3}{x_1 + x_3} \qquad p_{31} = 0$$

$$p_{12} = \frac{x_3}{x_2 + x_3} \qquad p_{22} = \frac{x_1}{x_1 + x_3} \qquad p_{32} = \frac{x_2}{x_1 + x_2}$$

$$p_{13} = \frac{x_2}{x_2 + x_3} \qquad p_{23} = p_{33} = 0$$

In our theorem we exclude sources/sinks in the graph G(_).

Note that, in general, $p_{ij}$ j=1 …. k is not a distribution.

What happens when $p_{ij}$ j=1 …k is indeed a prob. distribution?

## An answer

Specs of SPP independent of time, (and of the configuration)

Let, for all i

$$\_{i}(x(t)) = \_{i}$$

and for all i, j

$$p_{ij}(x(t)) = p_{ij}$$

Then the dynamics of SPP become

$$x\&_{i} \sum_{j \in k} x_{j} \ddot{e}_{j} p_{ji} - \ddot{e}_{i} x_{i}$$

i = 1 …k

**Definition**: The Markovian Population Protocols are SPP with ${\_}_{i}$, $p_{ij}$ independent of time and of the configuration x(t).

Let us define “rates” $r_{ij}$ such that

**(1)** $i \neq j$ $\quad r_{ij} = {\_}_{i} p_{ij}$ for all i,j,
**(2)** $i = j$ $\quad r_{ii} = {\_}_{i}(p_{ii} - 1)$

Then (i = 1 … k)

(*) $$\dot{x}_i(t) = r_{ij} x_i(t) + \sum_{k \neq i} r_{ki} x_k(t)$$

Note that x(t) is such that _$_{xi}$(t) = 1 and can be seen as a prob. distribution itself!

The equations (*) are the dynamics of a Markov Chain of _ states and continuous time rates $q_{ij}$.

(This holds even when _$_i$, $p_{ij}$ are only functions of time!)

When all $r_{ij}$ are nonzero then this Markov Chain is **irreducible** and **homogeneous**.

Then, we conclude:

- The $\lim_{t \to \infty} x_i(t)$ always exist and are independent of x(0).
- The limiting distribution is **unique**, and is the solution of the system

$$r_{ii} x_i + \sum_{i \neq j} r_{ij} x_i = 0$$

- The faithful PPP is always stable.

## What is the gain of proving that a PPP is faithful?

## What is the gain?

Powerful analytic tools for asymptotic (in)stability
(Can be **decided**)

Let $\dot{x}_i = f_i(x)$ i = 1… _

Let x* = a solution of $f_i(x)=0$ i = 1…_

Let L = [$L_{ij}$]

Where $L_{ij} = \frac{\partial f_i}{\partial x_j}(x^*)$ (Jacobian)

Let an eigenvalue g of L be

g = a+iw

## Theorem [Hartman, 60]

At x*

**(1)** a<0 ∀g then x(t) → x*

**(2)** ∃g : a>0 then x(t) **diverges**

**(3)** ∃g _=0 then oscillations

i.e. in our case, if PPP dynamic can be written via differential equations, then we can decide **stability** in polynomial time.

# A study of a special case of faithful SPP, PPP.

# Linear Viral Protocols (LVP)

Assume that the rules of _ basically specify that agents adopt the state of “the first person they meet in the street” (if they adopt at all).

Formally, for all i, j, x(t)

$p_{ij}(x(t)) = x_j(t)$ (**)

So, the dynamics are

$$\dot{x}_i = \sum_{j \in E} x_j x_i \lambda_j(x) - x_i \lambda_i(x)$$

We now propose a **linear** model to capture the “immunity” that an agent (in a state) has against other agents in the population.

One can imagine **immunity** to be a measure of the degree of protection of agents when they interact.

# Assumptions (A1)

- We measure the immunity of an interacting $(q_i, q_j)$ pair by an integer $a_{ij}$.
- We require $a_{ji}=a_{ij}$
  (symmetry)
  thus

**Definition**: Let $A = [a_{ij}]$ be a symmetric matrix of integers.
The **immunity** of an agent in state $q_i$ is

$$t_i(x(t)) = a_{i1}x_1(t) + \cdots + a_{ik}x_k(t)$$

It is natural to assume that agents in state $q_i$ wish to review their states more often when their immunity is low.

So we assume
(Assumption A2)

For state $q_i$,

_$_i$(x(t)) = _-_$t_i$(x(t))

where _,_, reals, _>0, and
$t_i$(x)≤_/_ always.

**Definition** (Average immunity of a population)

$$t(x) = \sum_i x_i t_i(x)$$

**Definition**: The SPP with assumptions A1, A2 are called Linear Viral Protocols (LVP).

**Lemma E.1.**

The dynamics of LVP are

$$\dot{x}_i = \ddot{a}(t_i(x) - t(x))x_i$$

i.e. they are a _-rescaling of the **Replicator Dynamics** of evolutionary game theory.

So,

**Lemma E.2.**

The dynamics of LVP are equivalent to the Lotka-Volterra dynamics.

Note, from Lemma E.1 that the "rest" points $x^*$ of the dynamics (i.e. when $\dot{x}(t) = 0$ ) must satisfy

$$t_i(x^*) = t(x^*)$$

**Lemma E.3**.
When (if) LVP stabilizes, then the immunity of any agent is the average system immunity.

# Very Immune Rest Points

**Definition**

Let x* be a rest point of LVP.

Then x* **is very immune** iff, for any x(t) in a region around x* it holds:

$$t(x) < \sum_{i=1}^{k} x_i^* t_i(x)$$

It is intuitive to expect that the very immune rest points are stable.

To show this, we recall:

## Lyapunov's Theorem

Let $\dot{x} = f(x)$ be a time-independent nonlinear equation.

Let $V : \{x(t)\} \rightarrow R$ be continuously differentiable. Then if in a region around x* we have $\dot{V} < 0$ then x* is stable.

**V(x) is called a Lyapunov Potential.**

## Lemma E.4.

Very Immune Rest Points are stable.

## Proof

Consider the relative entropy

$$E(x) = -\sum_{i=1}^{k} x_i^* \ln\left(\frac{x_i}{x_i^*}\right) \qquad \text{(clearly } E(x^*)=0$$

We can easily show that $\dot{E}(x) < 0$ in the region around x* (region defined by the property “Very Immune”).

QED

# Conclusions

**(1)** In many cases, PPP can be studied as systems of time –independent nonlinear 2nd degree diff. equations.

**(2)** This can be used to efficiently decide stability of PPP

**(3)** We can "globally specify" PPP by the Switching Protocols approach (since SPP $\supseteq$ PPP when PPP are faithful).

**(4)** Faithful PPP and SPP contain several well-known dynamic systems!

**Thank you!**