

## Correction examen "Graphs and Algorithms"

February 12, 2008

### Part I. Cutwidth and search games

**Question 1.** Let  $Y \subseteq X$ . We have :

$$2f(Y) = f(Y) + f(Y) = f(Y) + f(X \setminus Y) \geq f(X) + f(\emptyset) = 2f(\emptyset)$$

**Question 2.** The function  $f$  satisfies the first condition of the connectivity function because it is symmetric in  $S$  and  $V(G) \setminus S$ .

Let  $e = \{u, v\}$  with  $u \in Y_1 \cup Y_2$  and  $v \in V(G) \setminus (Y_1 \cup Y_2)$ . Then  $u \in Y_i$  for  $i = 1$  or  $2$  or both. Since  $v \in V(G) \setminus (Y_1 \cup Y_2)$ , we have  $v \in V(G) \setminus Y_i$ . Therefore, if the weight of  $e$  contributes to  $f(Y_1 \cup Y_2)$ , it also contributes to  $f(Y_1) + f(Y_2)$ .

Let  $e = \{u, v\}$  with  $u \in Y_1 \cap Y_2$  and  $v \in V(G) \setminus (Y_1 \cap Y_2)$ . Then  $u \in Y_i$  for  $i = 1$  and  $2$ . Since  $v \in V(G) \setminus (Y_1 \cap Y_2)$ , we have  $v \notin Y_i$  for  $i = 1$  or  $2$  or both. Thus  $v \in V(G) \setminus Y_i$ . Therefore, if the weight of  $e$  contributes to  $f(Y_1 \cap Y_2)$ , it also contributes to  $f(Y_1) + f(Y_2)$ .

If the weight of  $e$  contributes to both  $f(Y_1 \cup Y_2)$  and  $f(Y_1 \cap Y_2)$ , then  $v \notin (Y_1 \cup Y_2)$ , and thus  $e$  contributes to both  $f(Y_1)$  and  $f(Y_2)$ .

If some weights are negative, then the inequality can be violated. For instance, consider an edge  $e = \{u, v\}$  with  $u \in Y_1 \setminus Y_2$  and  $v \in Y_2 \setminus Y_1$ . This edge does not contribute to  $f(Y_1 \cup Y_2) + f(Y_1 \cap Y_2)$ , but it contributes to both  $f(Y_1)$  and  $f(Y_2)$ . As a consequence, if its weight is negative, the right hand side of the inequality may be smaller than the left hand side.

**Question 3.** The proof follows the guidelines of the monotony proof of Seymour and Thomas for crusades which was presented during the lectures. Let  $Z = (Y_1, \dots, Y_r)$  be a  $(k, \ell)$ -expansion in  $X$  satisfying :

C1 :  $\sum_{i=1}^r f(Y_i)$  is minimum ;

C2 :  $\sum_{i=1}^r |Y_i|$  is minimum subject to C1.

We prove that this expansion is monotone.

First, for any  $i$ , we have  $f(Y_{i-1} \cup Y_i) \geq f(Y_i)$ . Indeed, otherwise, let

$$Z' = (Y_1, \dots, Y_{i-1}, Y_{i-1} \cup Y_i, Y_{i+1}, \dots, Y_r).$$

$Z'$  satisfies conditions 1 and 2 of  $(k, \ell)$ -expansion. Moreover :

– We have  $(Y_{i-1} \cup Y_i) \setminus Y_{i-1} = Y_i \setminus Y_{i-1}$ , thus  $|(Y_{i-1} \cup Y_i) \setminus Y_{i-1}| = |Y_i \setminus Y_{i-1}| \leq \ell$ .

– We also have  $Y_{i+1} \setminus (Y_{i-1} \cup Y_i) \subseteq Y_{i+1} \setminus Y_{i-1}$ , thus  $|Y_{i+1} \setminus (Y_{i-1} \cup Y_i)| \leq |Y_{i+1} \setminus Y_{i-1}| \leq \ell$ .

Therefore,  $Z'$  also satisfies condition 3 of  $(k, \ell)$ -expansion. Thus if  $f(Y_{i-1} \cup Y_i) < f(Y_i)$  then we would get  $Z'$  is a  $(k, \ell)$ -expansion, in contradiction with C1.

Now, because of condition 2 of connectivity function,  $f(Y_{i-1} \cup Y_i) \geq f(Y_i)$  implies that  $f(Y_{i-1} \cap Y_i) \leq f(Y_{i-1})$ . Let

$$Z'' = (Y_1, \dots, Y_{i-2}, Y_{i-1} \cap Y_i, Y_i, \dots, Y_r).$$

$Z''$  satisfies conditions 1 and 2 of  $(k, \ell)$ -expansion. Moreover :

- We have  $(Y_{i-1} \cap Y_i) \setminus Y_{i-2} \subseteq Y_{i-1} \setminus Y_{i-2}$ , thus  $|(Y_{i-1} \cap Y_i) \setminus Y_{i-2}| \leq |Y_{i-1} \setminus Y_{i-2}| \leq \ell$ .
- We also have  $Y_i \setminus (Y_{i-1} \cap Y_i) = Y_i \setminus Y_{i-1}$ , thus  $|Y_i \setminus (Y_{i-1} \cap Y_i)| = |Y_i \setminus Y_{i-1}| \leq \ell$ .

Therefore,  $Z''$  also satisfies condition 3 of  $(k, \ell)$ -expansion. Thus,  $Z''$  is a  $(k, \ell)$ -expansion. It is minimal according to C1. Therefore, because of C2,  $|Y_i \cap Y_{i-1}| \geq |Y_{i-1}|$ , and thus  $Y_{i-1} \subseteq Y_i$ .

**Question 4.** Assume that there exists a  $(k, \ell)$ -expansion  $Z = (Y_1, \dots, Y_r)$  in  $X$ . Note that since  $f(Y_{i-1}) \leq k$  and  $f(Y_i) \leq k$ , we have at least one inequality  $f(Y_{i-1} \cap Y_i) \leq k$  or  $f(Y_{i-1} \cup Y_i) \leq k$  that is satisfied. Therefore, one can define the search following search strategy  $S = (M_1, M'_1, M_2, M'_2, \dots, M_r, M'_r)$  where

$$(M_i, M'_i) = \begin{cases} ((Y_{i-1} \setminus Y_i, 0), (Y_i \setminus Y_{i-1}, 1)) & \text{if } f(Y_{i-1} \cap Y_i) \leq k \\ ((Y_i \setminus Y_{i-1}, 1), (Y_{i-1} \setminus Y_i, 0)) & \text{if } f(Y_{i-1} \cup Y_i) \leq k \end{cases} \quad (1)$$

By construction, the cost of this strategy is at most  $k$ , and the its rate is at most  $\ell$ .

Conversaly, let  $S = (M_1, \dots, M_r)$  with  $M_i = (Y_i, b_i)$ , be a winning search strategy in  $X$  with rate  $\leq \ell$  and cost  $\leq k$ , with corresponding search sequence  $T = (N_0, \dots, N_r)$ . This latter sequence is a  $(k, \ell)$ -expansion. Indeed, by construction,  $N_0 = \emptyset$ ,  $N_r = X$ , and  $f(N_i) \leq k$ . For the last condition, we have :

If  $b_i = 0$ , then  $N_i = N_{i-1} \setminus Y_i$ , and thus  $|N_i \setminus N_{i-1}| \leq |Y_i| \leq \ell$ ;

If  $b_i = 1$ , then  $N_i = N_{i-1} \cup Y_i$ , and thus  $|N_i \setminus N_{i-1}| \leq |Y_i| \leq \ell$ .

**Question 5.** Assume that there exists a winning search strategy with rate  $\ell$  and cost  $k$ .

Then by Question 4, there exists a  $(k, \ell)$ -expansion.

Then, by Question 3, there exists a monotone  $(k, \ell)$ -expansion  $Y_1, \dots, Y_r$ .

Then, by Question 4, there exists a monotone winning search strategy with rate  $\leq \ell$  and cost  $\leq k$ . Indeed, the construction of Equation 1 preserves monotony. In other words, the search strategy  $S = (M_1, \dots, M_r)$  with  $M_i = (Y_i \setminus Y_{i-1}, 1)$  is a monotone search strategy.

**Question 6.** Consider the node search version of graph searching, excepted that if  $d$  edges incident to a clear node lead to the contaminated part of the graph, then  $d$  searchers are required to guard that node. The connectivity function  $f$  for that game is the one defined in Question 2. Since searcher are added or removed one by one in node search, the rate of that game is 1. Let  $\text{ns}(G)$  be the minimum number of searchers required to clear the graph in this game.

By Question 5, the game is monotone, that is : if there exists a search strategy with cost at most  $k$ , then there exists a monotone search strategy with cost at most  $k$ . Now, a monotone search strategy for that game is a sequence of nodes  $x_1, \dots, x_n$  where  $x_i$  is the node added at step  $i$  of the strategy. Since the cost for guarding  $S_i$  is  $f(S_i)$  we get a one-to-one correspondence between layouts of width at most  $k$  and monotone search strategies of cost at most  $k$ . Therefore  $\text{cw}(G) = \text{ns}(G)$ .

## Part II. Routing and Cayley graphs

**Question 1.** If all routes  $R_{u,v}$  are along shortest paths, then the total load of the routes is

$$L = \sum_{u \in V(G)} \sum_{v \in V(G), v \neq u} (\text{dist}_G(u, v) - 1).$$

If the number of paths traversing a node is the same for all nodes, it means that this load is evenly shared by the nodes, and thus any node has a load  $L/n$ .

**Question 2.** The graph (a) in Figure 1 is not a Cayley graph because it is even not vertex-transitive. Indeed, the number of nodes at distance 2 is not the same for the grey nodes and the white nodes.

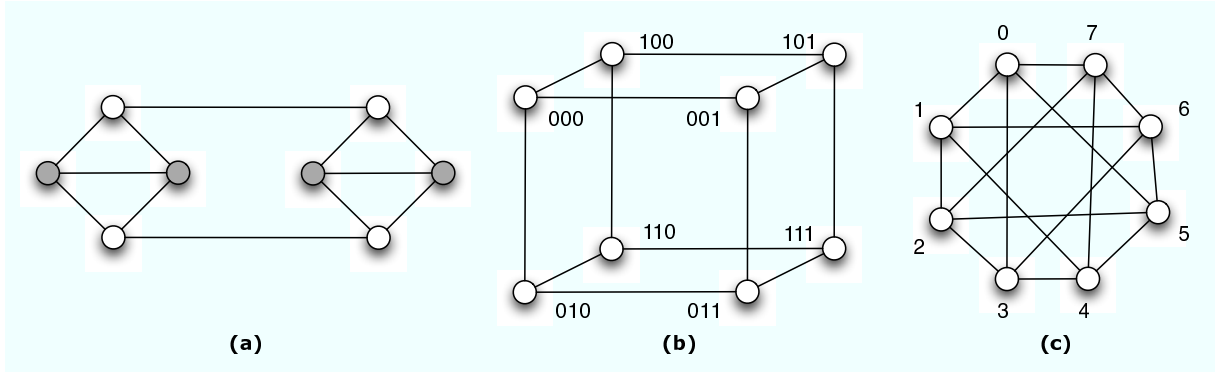


FIG. 1 – (a) is not Cayley, (b) and (c) are Cayley

The graph (b) is the 3-dimensional hypercube. It is a Cayley graph  $(\Gamma, S)$  with  $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $S = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$ .

The graph (c) is a chordal graph. It is a Cayley graph  $(\Gamma, S)$  with  $\Gamma = \mathbb{Z}/8\mathbb{Z}$  and  $S = \{+1, +3, +5, +7\}$ .

**Question 3.** Important : note that, for any  $u \in \Gamma$ , the map  $\phi_u : v \mapsto u * v$  is an automorphism of the Cayley graph  $G = (\Gamma, S)$ .

The routes  $R_{e,v}$  as shortest path by construction. For  $u \neq e$ , and  $u \neq v$ ,  $R_{u,v} = u * R_{e, u^{-1} * v}$ . Let  $R_{e, u^{-1} * v} = (x_0, x_1, \dots, x_k)$  with  $x_0 = e$  and  $x_k = u^{-1} * v$ . We get  $R_{u,v} = (u * x_0, u * x_1, \dots, u * x_k)$ . Assume that there is a shorter path  $P = (y_0, y_1, \dots, y_\ell)$  from  $u$  to  $v$ , i.e.,  $y_0 = u$ ,  $y_\ell = v$ , and  $\ell < k$ . Then  $u^{-1} * P$  is a path from  $e$  to  $u^{-1} * v$ , of length  $\ell < k$ , contradicting the fact that  $R_{e, u^{-1} * v}$  is a shortest path.

**Question 4.** Let  $x$  and  $x'$  be two different nodes of  $G$ . Assume that  $x$  is traversed by the route  $R_{u,v}$ . Let  $y = x' * x^{-1}$ . The map  $\phi_y$  maps  $x$  to  $x'$ . Let  $u' = \phi_y(u)$  and  $v' = \phi_y(v)$ . We have

$$R_{u',v'} = u' * R_{e, u'^{-1} * v'} = x' * x^{-1} * u * R_{e, (x' * x^{-1} * u)^{-1} * (x' * x^{-1} * v)} = x' * x^{-1} * u * R_{e, u^{-1} * v} = \phi_y(R_{u,v}).$$

**Question 5.** From Question 3, all routes are along shortest paths. Question 4 shows that there is a one-to-one mapping from the routes traversing a node  $x$  to the routes traversing any other node  $x'$ . Hence the load of every node is the same. Thus, by Question 1,

$$\xi(G, R, x) = \frac{1}{n} \sum_{u \in V(G)} \sum_{v \in V(G), v \neq u} (\text{dist}_G(u, v) - 1).$$

Since, Cayley graphs are vertex-transitive, we get

$$\xi(G, R, x) = \sum_{v \in V(G), v \neq u_0} (\text{dist}_G(u_0, v) - 1)$$

for any node  $u_0 \in V(G)$ . Since, this is also a lower bound for any routing, we get the result.

### Part III. Bandwidth and broadcasting

**Question 1.** Let the star of  $n$  nodes be  $S_n$  with  $V(S_n) = \{x_0, x_1, \dots, x_{n-1}$  where  $x_0$  is the center. Let  $k \geq 1$ . Consider the following path-decomposition  $P(k) = X_1, \dots, X_{\lceil (n-1)/k \rceil}$  where

$$X_i = \{x_0\} \cup \{x_{k(i-1)+j}, j = 1, \dots, k\}.$$

The pathwidth of this path-decomposition is  $\max_i |X_i| - 1 = k$ . The center appears in all bags, hence the bandwidth of this path-decomposition is  $\lceil (n-1)/k \rceil$ .

**Question 2.** Additional hypotheses were required for the result to hold. Any attempt to solve that question was acknowledged by additional points according to the degree of advance in the solution. The global reckoner does not include this question.

**Question 3.** Consider the following path-decomposition  $P$  of an  $n \times m$  grid with node set  $\{x_{i,j}, i = 1, \dots, m, j = 1, \dots, n\}$ .  $P = X_1, \dots, X_{n-1}$  where  $X_j = \{x_{i,j}, x_{i,j+1}, i = 1, \dots, m\}$ . Its pathwidth is  $\omega = 2m - 1$ , and its bandwidth is  $\beta = 2$ . Following Question 3,  $b(G) \leq O(\log n)$ . On the other hand, since a node can send the information to at most one other node at each round, we have  $b(G) \leq \lceil \log_2 n \rceil$ .

— End —