— MPRI —
Examen "Graphs and Algorithms"
February 12, 2008 — 9h00-12h00

Part I. Cutwidth and search games

Let X be a finite set. A function $f : \mathcal{P}(X) \to \mathbb{N}$ is a *connectivity function* if the following two conditions are satisfied :

1. $\forall Y \subseteq X,\ \ f(Y) = f(X \setminus Y)$;
2. $\forall Y_1, Y_2 \subseteq X,\ \ f(Y_1 \cup Y_2) + f(Y_1 \cap Y_2) \leq f(Y_1) + f(Y_2)$;

Question 1. Prove that if f is a connectivity function then $f(\emptyset) \leq f(Y)$ for any $Y \subseteq X$.

Question 2. Let (G, ω) be an edge-weighted graph. Let f be the function that assign to any subset $S \subseteq V(G)$ the sum of the weights $\omega(e)$ of all edges e with one extremity in S and the other in $V(G) \setminus S$. Prove that if all weights are non negative, then f is a connectivity function, and explain what may happen if some weights are negative.

For any two positive integers k and ℓ, a sequence $(Y_1, \ldots, Y_r)$ of subsets of X is a (k, ℓ)-expansion in X if the following three conditions are satisfied :

1. $Y_1 = \emptyset$ and $Y_r = X$;
2. $\forall i \in \{1, \ldots, r\},\ \ f(Y_i) \leq k$;
3. $\forall i \in \{1, \ldots, r-1\},\ \ |Y_{i+1} \setminus Y_i| \leq \ell$;

If we also have that, for any $i \in \{1, \ldots, r-1\}$, $Y_i \subseteq Y_{i+1}$ then the sequence is said *monotone*.

Question 3. Prove that if there is a (k, ℓ)-expansion in X then there is a monotone (k, ℓ)-expansion in X.

Let X be a finite set and f a connectivity function in X. We now define a search game for (X, f). A *move* is a pair (Y, b) where $Y \subseteq X$ and b is a boolean : $b = 1$ means that a mark is placed on every element of Y, while $b = 0$ means that the mark of every element of Y is removed. A search *strategy* is a sequence $S = (M_1, \ldots, M_r)$ of moves, with $M_i = (Y_i, b_i)$. The corresponding search *sequence* is $T = (N_0, \ldots, N_r)$ where $N_0 = \emptyset$, and, for $i > 0$, $N_i = N_{i-1} \cup Y_i$ if $b_i = 1$, and $N_i = N_{i-1} \setminus Y_i$ if $b_i = 0$. A winning search strategy is a strategy $S = (M_1, \ldots, M_r)$ for which the search sequence $T = (N_0, \ldots, N_r)$ satisfies $N_r = X$. A strategy is monotone if it does not contain any move (Y, b) with $b = 0$ (i.e., a mark is never removed).

The *rate* of a search strategy $S = (M_1, \ldots, M_r)$ is $\max_{\{i | b_i = 1\}} |Y_i|$, and its *cost* is $\max_i f(N_i)$.

Question 4. Prove that there exists a (k, ℓ)-expansion in X if and only if there exists a winning search strategy in X with rate $\leq \ell$ and cost $\leq k$.

Question 5. Prove that if there exists a winning search strategy with rate ℓ and cost k then there exists a monotone winning search strategy with rate $\leq \ell$ and cost $\leq k$.

The *cutwidth* of a graph is notion that is used extensively in VLSI design for processor architecture. Let G be an n-node graph, and let f be the function that assign to any subset $S \subseteq V(G)$ the number of edges with one extremity in S and the other in $V(G) \setminus S$. A *layout* $\mathcal{L}$ is an ordering $(x_1, \ldots, x_n)$ of the nodes of G. For $i \in \{1, \ldots, n\}$, let $S_i = \{x_1, \ldots, x_i\}$, and let $\text{width}(G, \mathcal{L}) = \max_{i=1,\ldots,n} f(S_i)$. The cutwidth of G is defined as

$$\text{cw}(G) \; = \; \min_{\text{layouts } \mathcal{L}} \text{width}(G, \mathcal{L})$$

Question 6. Define the cutwidth in term of search game.

Part II. Routing and Cayley graphs

Recall that a routing in an n-node graph G is a set $R = \{R_{u,v}, u \in V(G), v \in V(G), u \neq v\}$ where $R_{u,v}$ denotes a path from u to v in G.

Question 1. Prove that if a routing R satisfies (1) all routes $R_{u,v}$ are along shortest paths, and (2) the number of paths traversing a node is the same for all nodes, then the load $\xi(G, R, x)$ of any node x satisfies

$$\xi(G, R, x) = \frac{1}{n} \sum_{u \in V(G)} \; \sum_{v \in V(G), v \neq u} (\text{dist}_G(u, v) - 1)$$

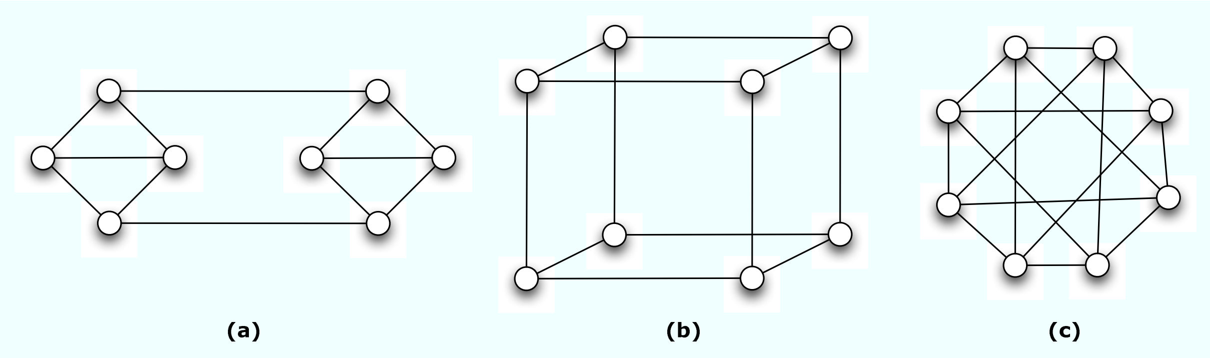
Recall that a graph G is a Cayley graph if there exist a group $(\Gamma, *)$, and a generating set S of Γ with $S^{-1} = S$, such that

$$V(G) = \Gamma$$

and

$$\{u, v\} \in E(G) \iff u^{-1} * v \in S.$$

Question 2. Tell which graphs in the figure below are Cayley, and which ones are not. Explain why.



Let us consider the following routing R in a Cayley graph G. Let e be the neutral element of Γ. For any $v \neq e$, we define $R_{e,v}$ as one shortest path in G from e to v, and, for $u \neq e$, we define $R_{u,v} = u * R_{e,u^{-1}*v}$.

Question 3. Prove that all routes $R_{u,v}$ are along shortest paths.

Question 4. Let x and x' be two different nodes of G. Assume that x is traversed by the route $R_{u,v}$. Prove that there exists two nodes u' and v' such that x' is traversed by the route $R_{u',v'}$.

Let $\xi(G) = \min_R \max_x \xi(G, R, x)$ be the load of a routing of minimal load.

Question 5. Prove that $\xi(G) = \sum_{v \neq u_0}(\mathrm{dist}_G(u_0, v) - 1)$ for any node $u_0 \in V(G)$.

Part III. Bandwidth and broadcasting

We define the *bandwidth* of a path-decomposition $P = (X_1, \ldots, X_k)$ of a graph G as

$$\mathrm{bw}(P) = 1 + \max_{u \in V(G)} (r_u - \ell_u)$$

where r_u and ℓ_u are the indices such that $u \in X_i$ if and only if $i \in [\ell_u, r_u]$.

Question 1. Prove that the star (i.e., the n-node tree with one centre connected to $n-1$ leaves) has a path decomposition of pathwidth k and bandwidth $\lceil (n-1)/k \rceil$ for any $2 \leq k \leq n$.

We consider the broadcast problem under the following model : (1) a node can send a message to at most one other node at every round, along a path of arbitrary length, (2) the communication paths used at the same round must be pairwise node-disjoint. We denote by $b(G, s)$ the number of rounds that is necessary and sufficient to achieve broadcasting of one message in G from source s. Let $b(G) = \max_{s \in V(G)} b(G, s)$.

Question 2. Prove that if G has a path-decomposition with N bags, pathwidth ω, and bandwidth β, then $b(G) \leq O(\omega\beta + \log_2(N/\beta))$.
`WARNING : additional hypotheses are required for this to hold. Skip that question, and just assume that it is true for solving the next question.`

Question 3. Prove that the result of Question 2 is asymptotically optimal for $n \times m$ grids where m is any constant, and $n \to +\infty$.

— *End* —