

## Examen "Graphs and Algorithms"

February 10, 2009 — 08h45-10h15

No electronic communication devices (e.g., portable computer, mobile phone, etc.) are allowed. Lecture notes (paper) are however authorized.

### Exercise I. Multicast

Let  $G$  be a strongly connected directed graph,  $D \subseteq V(G)$ , and  $s \in D$ . We consider the 1-port communication model in which, at each round, nodes communicate by pairs along simple paths of arbitrary length. Communication paths used at the same round must be pairwise node-disjoint.

During the lecture, we have seen that, unless  $P=NP$ , the multicast time  $b(G, D, s)$  from  $s$  to  $D$  in  $G$  cannot be approximated in polynomial time within less than factor  $\Omega(\log |D|)$ . In the first part of this exercise, we will see that, despite this negative result, one can achieve good approximation of the broadcast problem, i.e., the multicast problem where  $D = V(G)$ .

The tree considered in this exercise are rooted, and have their edges directed towards their leaves.

A  $(\rho, r)$ -approximation algorithm for the Minimum Degree Spanning Tree (MDST) problem in digraphs is an algorithm that, given any strongly connected digraph  $G$  and any  $D \subseteq V(G)$ , returns a tree spanning  $D$ , and of maximum out-degree at most  $\rho \Delta_{min}^+(G, D) + r$ , where  $\Delta_{min}^+(G, D)$  is the minimum, taken over all trees  $T$  spanning  $D$  in  $G$ , of the out-degree of  $T$ .

**Question 1.** Prove that  $b(G, D, s) \geq \max\{\log |D|, \Delta_{min}^+(G, D)\}$ .

Let  $T^*$  be a rooted tree spanning  $D$  in  $G$  and of maximum out-degree at most  $\Delta^+$ .

**Question 2.** Let  $m \in \{1, \dots, |D|\}$ . Prove that there exists  $D^* \subset D$  with  $|D^*| \leq |D|/m$ , and  $s^* \in V(G)$  such that  $b(G, D, s) \leq 2m + \lceil \log |D| \rceil + 2\Delta^+ + b(T^*, D^* \cup \{s^*\}, s^*)$ .

**Question 3.** Assume that there exists an algorithm  $\mathcal{A}$ , running in time  $t(n)$ , which returns, for any  $n$ -node digraph  $G$ , a  $(O(1), O(\log n))$ -approximation algorithm for the MDST problem restricted to  $D = V(G)$ . Design an  $O(\frac{\log n}{\log \log n})$ -approximation algorithm  $\mathcal{B}$  for the broadcast problem, running in time  $t(n) + O(n)$ .

**Question 4.** Prove that, unless  $P = NP$ , any polynomial-time  $(O(1), r)$ -approximation algorithm for the general version of the MDST problem (i.e., arbitrary  $D$ ) in directed graphs satisfies  $r = \Omega(\log |D| \log \log |D|)$ .

## Exercise II. Informative labeling

Let  $\mathcal{G}$  be a set of graphs. A distance-labeling scheme for  $\mathcal{G}$  is defined by a marker-decoder pair  $(M, D)$  where  $M$  labels the nodes of any  $G \in \mathcal{G}$  by binary strings in  $\{0, 1\}^*$  and  $D : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \mathbb{N}$ . This pair must satisfy, for any  $G \in \mathcal{G}$ , for any  $(u, v) \in V(G) \times V(G)$ ,

$$D(M(G, u), M(G, v)) = \text{dist}_G(u, v).$$

First, we focus on distance-labeling for trees.

**Question 1.** Let  $c$  be a center of a tree  $T$ . We say that two nodes  $u$  and  $v$  of two distinct subtrees of  $T$  obtained by removing  $c$  from  $T$  are separated. Compute a partial distance-labeling scheme that satisfies the definition of distance-labeling whenever  $u$  and  $v$  are separated. What is the size of the labels used by this partial distance-labeling scheme?

**Question 2.** Compute a distance-labeling scheme for trees using labels of  $O(\log^2 n)$  bits in  $n$ -node trees.

Let's now move on to arbitrary graphs.

**Question 3.** Prove that there is a distance-labeling scheme for  $n$ -node graphs of diameter at most  $D$  using label on  $O(n \log D)$  bits.

**Question 4.** Extend the scheme designed in Question 2 to get a distance-labeling scheme for  $n$ -node graphs of treewidth at most  $k$  using label on  $O(k \log^2 n)$  bits.

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