Numbers with integer expansion in the numeration system with negative base

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joint work with D. Dombek, Z. Masáková, and E. Pelantová

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Digital expansions, dynamics and tilings
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Outline

β-expansions of real numbers

Consider $\beta > 1$ and $T_\beta : [0,1) \mapsto [0,1)$ given by

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor \,.$$

Representation of $x \in [0,1)$ of the form

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots \,,$$

where

$$x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$$

is called the **β-expansion** of x.

We write

$$\mathrm{d}_\beta(x) := x_1 x_2 x_3 \cdots \,.$$

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The β-expansion of $x \geq 1$ can be naturally defined:

- find an exponent $k \in \mathbb{N}$ such that $\frac{x}{\beta^k} \in [0,1)$
- using the transformation T_β derive the β-expansion of $\frac{x}{\beta^k}$

$$\frac{x}{\beta^k} = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots ,$$

- then

$$x = x_1\beta^{k-1} + x_2\beta^{k-2} + \cdots + x_{k-1}\beta + x_k + \frac{x_{k+1}}{\beta} + \cdots .$$

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An integer sequence

$$x_1 x_2 x_3 \cdots$$

is said to be **β-admissible** if there exists $x \in [0,1)$

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Set of non-negative **β-integers** is

$$\mathbb{Z}_\beta^+ := \{x_k \beta^k + \cdots + x_1 \beta + x_0 \mid x_k \cdots x_0 0^\omega \text{ is a } \beta\text{-admissible sequence}\} .$$

[Thurston]: The distances between consecutive β-integers take values in $\{\Delta_i \mid i = 0, 1, 2 \cdots\}$, where

$$\Delta_i = \sum_{j=1}^{\infty} \frac{t_{i+j}}{\beta^j} \qquad \text{and} \qquad d_\beta(1) = t_1 t_2 t_3 \cdots .$$

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Theorem (Ito-Sadahiro)

The string $x_1x_2x_3\cdots$ is $(-\beta)$-admissible, if and only if for all $i = 1, 2, 3, \dots,$

$$\mathrm{d}_{-\beta}(l_\beta) \preceq_{alt} x_i x_{i+1} x_{i+2} \prec_{alt} \mathrm{d}^*_{-\beta}(r_\beta),$$

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Alternate order.

$x_1x_2x_3\cdots \prec_{\text{alt}} y_1y_2y_3\cdots$
if $(-1)^i(x_i - y_i) > 0$ for the smallest i such that $x_i \neq y_i$.

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Relation between $\mathrm{d}^*_{-\beta}(r_\beta)$ and $\mathrm{d}_{-\beta}(l_\beta)$.

$$\mathrm{d}^*_{-\beta}(r_\beta) = \begin{cases} (0 l_1 \cdots l_{2l}(l_{2l+1} - 1))^\omega & \text{for } \mathrm{d}_{-\beta}(l_\beta) = (l_1 \cdots l_{2l+1})^\omega \\ 0\mathrm{d}_{-\beta}(l_\beta) & \textit{otherwise.} \end{cases}$$

Uniqueness problem

Consider $x = \frac{\beta^2}{\beta+1} \notin \mathrm{I}_\beta = \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$.

- $\frac{x}{-\beta} = \frac{-\beta}{\beta+1}$. Thus

$$\mathrm{d}_{-\beta}\left(\frac{x}{-\beta}\right) = l_1 l_2 l_3 \cdots$$

- $\frac{x}{(-\beta)^3} = \frac{1}{-\beta(\beta+1)} \in \mathrm{I}_\beta$. We compute

$$x_1 = \left\lfloor -\beta \frac{x}{(-\beta)^3} + \frac{\beta}{\beta+1} \right\rfloor = \left\lfloor \frac{1}{\beta+1} + \frac{\beta}{\beta+1} \right\rfloor = 1$$

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Uniqueness II

Lemma

Let $x_1x_2x_3\cdots$ be a $(-\beta)$-admissible sequence with $x_1 \neq 0$. For fixed $k \in \mathbb{Z}$, denote

$$z = \sum_{i=1}^{\infty} x_i(-\beta)^{k-i}\,.$$

Then

$$z \in \begin{cases} \left[\frac{\beta^{k-1}}{\beta+1}, \frac{\beta^{k+1}}{\beta+1}\right] & \text{for } k \text{ odd,} \\ \left[-\frac{\beta^{k+1}}{\beta+1}, -\frac{\beta^{k-1}}{\beta+1}\right] & \text{for } k \text{ even.} \end{cases}$$

Remark. Numbers with two different $(-\beta)$-admissible expansions

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Greedy algorithm

Greedy algorithm for computing the $(-\beta)$-expansion of $x \in \mathbb{R}$

Require: $x \in \mathbb{R}$
 while $x \neq 0$ **do**
  if $x > 0$ **then**
   find the maximal even $k \in \mathbb{Z}$ such that $x \geq \frac{\beta^k}{\beta+1}$.
  end if
  if $x < 0$ **then**
   find the maximal odd $k \in \mathbb{Z}$ such that $x \leq \frac{-\beta^k}{\beta+1}$.
  end if
  $x_k \leftarrow \lfloor \frac{x}{(-\beta)^k} + \frac{\beta}{\beta+1} \rfloor$
  $x \leftarrow x - x_k(-\beta)^k$
 end while

Outline

Set of **$(-\beta)$-integers**

$$\mathbb{Z}_{-\beta} = \{x_k(-\beta)^k + \cdots + x_1(-\beta) + x_0 \mid x_k \cdots x_1 x_0 0^\omega \text{ is } (-\beta)\text{-admissible}\}\,.$$

Remark.

- $0 \in \mathrm{I}_\beta$ and $T_{-\beta}(0) = 0 \Rightarrow \mathrm{d}_{-\beta}(0) = 0^\omega$ and thus $0 \in \mathbb{Z}_{-\beta}$
- β minimal Pisot number, then $\mathrm{d}_{-\beta}(l_\beta) = 1001^\omega$

  $x_k \cdots x_1 x_0 0^\omega \neq 0^\omega$ is $(-\beta)$-admissible $\Rightarrow$ so is 10^ω

  But $1001^\omega \not\preceq_{\text{alt}} 10^\omega \Rightarrow \mathbb{Z}_{-\beta} = \{0\}$.

Lemma

$\mathbb{Z}_{-\beta} = \{0\}$ *iff* $10^{2k}1$ *is a prefix of* $\mathrm{d}_{-\beta}(l_\beta)$ *for some* $k \geq 0$.

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$$\mathbb{Z}_{-\beta} = \{x_k(-\beta)^k + \cdots + x_1(-\beta) + x_0 \mid x_k \cdots x_1 x_0 0^\omega \text{ is } (-\beta)\text{-admissible}\}.$$

Remark.

- $0 \in \mathrm{I}_\beta$ and $T_{-\beta}(0) = 0 \Rightarrow \mathrm{d}_{-\beta}(0) = 0^\omega$ and thus $0 \in \mathbb{Z}_{-\beta}$
- β minimal Pisot number, then $\mathrm{d}_{-\beta}(l_\beta) = 1001^\omega$
  $x_k \cdots x_1 x_0 0^\omega \neq 0^\omega$ is $(-\beta)$-admissible $\Rightarrow$ so is 10^ω
  But $1001^\omega \not\preceq_{\text{alt}} 10^\omega \Rightarrow \mathbb{Z}_{-\beta} = \{0\}$.

Lemma

$\mathbb{Z}_{-\beta} = \{0\}$ *iff* $10^{2k}1$ *is a prefix of* $\mathrm{d}_{-\beta}(l_\beta)$ *for some* $k \geq 0$.

Gaps in $\mathbb{Z}_{-\beta}$ — general proposition

$$\mathcal{S}(k) = \{x_{k-1}x_{k-2}\cdots x_0 0^\omega \mid x_{k-1}x_{k-2}\cdots x_0 0^\omega \text{ is } (-\beta)\text{-admissible}\}\,.$$

$$\mathrm{Max}(k) = \text{maximal in } \mathcal{S}(k) \text{ with respect to the alternate order,}$$
$$\mathrm{Min}(k) = \text{minimal in } \mathcal{S}(k) \text{ with respect to the alternate order.}$$

Proposition

Let Δ be the distance of two consecutive $(-\beta)$-integers. Then there exists a $k \in \{0, 1, 2, \dots\}$ such that

$$\Delta = \beta^{2k} + \gamma(\mathrm{Min}(2k)) - \gamma(\mathrm{Max}(2k))$$

or

$$\Delta = \beta^{2k+1} + \gamma(\mathrm{Max}(2k+1)) - \gamma(\mathrm{Min}(2k+1))\,.$$

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Gaps in $\mathbb{Z}_{-\beta}$ — first case

Theorem

Let $\mathrm{d}_{-\beta}(l_\beta) = l_1 l_2 l_3 \cdots$ *where* $0 < l_i < l_1$ *for all* $i = 2, 3, 4, \ldots$. *Then the distances between adjacent* $(-\beta)$*-integers take values*

$$\Delta_0 = 1$$

$$\Delta_k = \left| (-1)^k + \sum_{i=1}^{\infty} \frac{l_{k-1+i} - l_{k+i}}{(-\beta)^i} \right|, \quad k = 1, 2, 3, \ldots$$

Moreover, all the distances are less than 2.

Gaps in $\mathbb{Z}_{-\beta}$ — second case

Theorem

Let $d_{-\beta}(l_\beta) = l_1 l_2 \cdots l_m 0^\omega$, *where* $l_m \neq 0$.
If $0 < l_i < l_1$ *for all* $i = 2, 3, 4, \dots, m$, *then*

m even
$$\begin{cases} \Delta_0 = 1\,, \\ \Delta_k = \left| (-1)^k + \sum_{i=1}^{\infty} \frac{l_{k-1+i} - l_{k+i}}{(-\beta)^i} \right|, \quad k = 1, \dots, m \end{cases}$$

m odd
$$\begin{cases} \Delta_0 = 1\,, \\ \Delta_k = \left| (-1)^k + \sum_{i=1}^{\infty} \frac{l_{k-1+i} - l_{k+i}}{(-\beta)^i} \right|, \quad k = 1, \dots, m-1\,, \\ \Delta_m = \frac{l_m}{\beta} \end{cases}$$

Moreover, all the distances are less than 2.

Outline

Tribonacci number β, root of $x^3 - x^2 - x - 1$

β-expansions

- $d_\beta(1) = 1110^\omega$
- $\Delta_0 = 1$, $\Delta_1 = \beta - 1$ and $\Delta_2 = \frac{1}{\beta}$.

$(-\beta)$-expansions

- $d_{-\beta}(l_\beta) = 101^\omega$,
- $d^*_{-\beta}(r_\beta) = 0101^\omega$,

$$\mathrm{Min}(2k) = 10(11)^{k-1}\,, \qquad \mathrm{Min}(2k+1) = 10(11)^{k-1}0\,,$$
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$$\mathrm{Max}(2) = 01\,.$$

Distances are

$$\Delta_0 = 1\,, \quad \Delta_1 = \beta - 1\,, \quad \text{and} \quad \Delta_2 = \frac{1}{\beta}\,.$$

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β root of $x^3 - 2x^2 - x + 1$

β-expansions

- $d_\beta(1) = 2(01)^\omega$
- $\Delta_0 = 1$, $\Delta_1 = \beta - 2$, and $\Delta_3 = \beta^2 - 2\beta$.

$(-\beta)$-integers

- $d_{-\beta}(l_\beta) = 210^\omega$
- By Theorem

$$\widetilde{\Delta}_0 = 1\,,$$
$$\widetilde{\Delta}_1 = \beta - 1\,,$$
$$\widetilde{\Delta}_2 = 1 - \frac{1}{\beta}\,.$$

Distances in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ are different.

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- By Theorem

$$\begin{aligned} \widetilde{\Delta}_0 &= 1\,, \\ \widetilde{\Delta}_1 &= \beta - 1 > 1\,, \\ \widetilde{\Delta}_2 &= 1 - \frac{1}{\beta}\,. \end{aligned}$$

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Outline

Open questions

- Gaps in $\mathbb{Z}_{-\beta}$ in general
- What does the projection of $\mathbb{Z}_{-\beta}$ into the contracting plane give?

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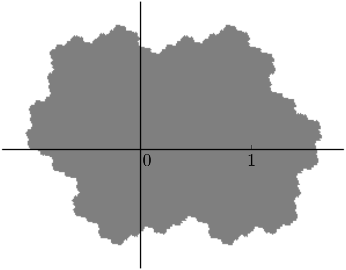
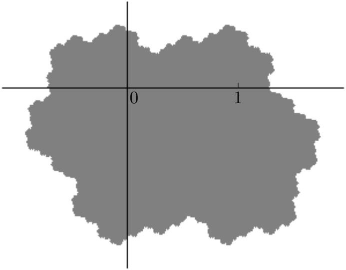


Projection of $\mathbb{Z}_{\beta}$,
β Tribonacci number.



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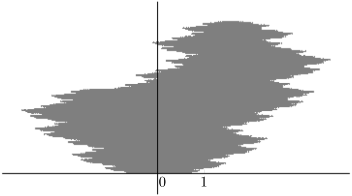

Projection of $\mathbb{Z}_{\beta}$,
β root of $x^3 = 2x^2 + x - 1$.

Projection of $\mathbb{Z}_{-\beta}$,
β root of $x^3 = 2x^2 + x - 1$.

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If we code gaps in $\mathbb{Z}_{-\beta}$ by an infinite word $\boldsymbol{u}_{-\beta}$

- Is $\boldsymbol{u}_{-\beta}$ fixed point of some substitution?
- Is there any relation with the canonical substitution φ_β associated to β-numeration system?

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