

ON THE STRUCTURE OF $(-\beta)$ -INTEGERS

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ABSTRACT. The $(-\beta)$ -integers are natural generalisations of the β -integers, and thus of the integers, for negative real bases. When β is the analogue of a Parry number, we describe the structure of the set of $(-\beta)$ -integers by a fixed point of an anti-morphism.

1. INTRODUCTION

The aim of this paper is to study the structure of the set of real numbers having a digital expansion of the form

$$\sum_{k=0}^{n-1} a_k (-\beta)^k,$$

where $(-\beta)$ is a negative real base with $\beta > 1$, the digits $a_k \in \mathbb{Z}$ satisfy certain conditions specified below, and $n \geq 0$. These numbers are called $(-\beta)$ -integers, and have been recently studied by Ambrož, Dombek, Masáková and Pelantová [1].

Before dealing with these numbers, we recall some facts about β -integers, which are the real numbers of the form

$$\pm \sum_{k=0}^{n-1} a_k \beta^k \quad \text{such that} \quad 0 \leq \sum_{k=0}^{m-1} a_k \beta^k < \beta^m \quad \text{for all } 1 \leq m \leq n,$$

i.e., $\sum_{k=0}^{n-1} a_k \beta^k$ is a greedy β -expansion. Equivalently, we can define the set of β -integers as

$$\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \quad \text{with} \quad \mathbb{Z}_\beta^+ = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0),$$

where T_β is the β -transformation, defined by

$$T_\beta : [0, 1) \rightarrow [0, 1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor.$$

This map and the corresponding β -expansions were first studied by Rényi [20].

The notion of β -integers was introduced in the domain of quasicrystallography, see for instance [6], and the structure of the β -integers is very well understood now. We have $\beta \mathbb{Z}_\beta \subseteq \mathbb{Z}_\beta$, the set of distances between consecutive elements of \mathbb{Z}_β is

$$\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\},$$

where $T_\beta^n(x^-) = \lim_{y \rightarrow x^-} T_\beta^n(y)$, and the sequence of distances between consecutive elements of \mathbb{Z}_β^+ is coded by the fixed point of a substitution, see [9] for the case when Δ_β is a finite set, that is when β is a Parry number. We give short proofs of these facts in Section 2. More detailed properties of this sequence can be found e.g. in [2, 3, 4, 11, 16].

Closely related to \mathbb{Z}_β^+ are the sets

$$S_\beta(x) = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(x) \quad (x \in [0, 1)),$$

which were used by Thurston [21] to define (fractal) tilings of \mathbb{R}^{d-1} when β is a Pisot number of degree d , i.e., a root > 1 of a polynomial $x^d + p_1 x^{d-1} + \dots + p_d \in \mathbb{Z}[x]$ such that all other roots have modulus < 1 , and an algebraic unit, i.e., $p_d = \pm 1$. These tilings allow e.g. to determine the k -th digit a_k of a number without knowing the other digits, see [15].

It is widely agreed that the greedy β -expansions are the natural representations of real numbers in a real base $\beta > 1$. For the case of negative bases, the situation is not so clear. Ito and Sadahiro [14] proposed recently to use the $(-\beta)$ -transformation defined by

$$T_{-\beta} : \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1} \right), x \mapsto -\beta x - \left\lfloor \frac{\beta}{\beta+1} - \beta x \right\rfloor,$$

see also [10]. This transformation has the important property that $T_{-\beta}(-x/\beta) = x$ for all $x \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1} \right)$. Some instances are depicted in Figures 1, 3, 4 and 5.

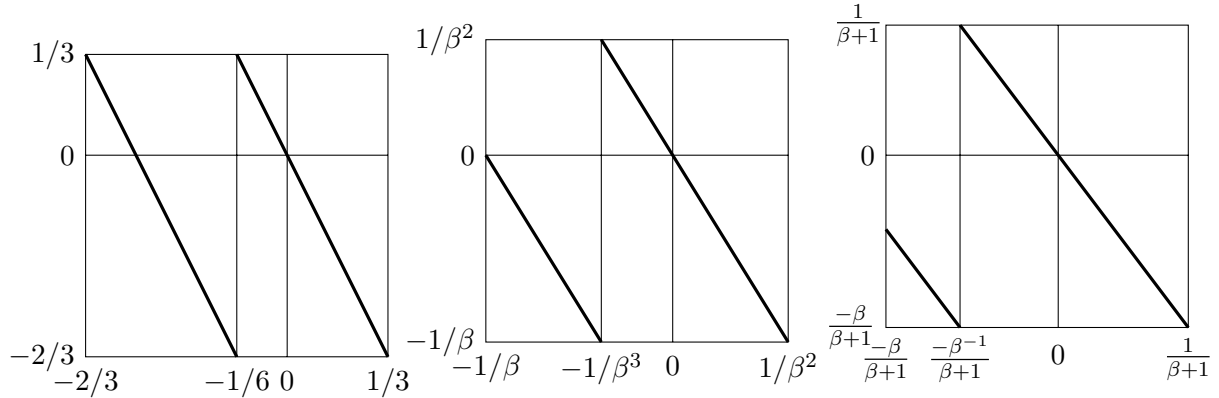


FIGURE 1. The $(-\beta)$ -transformation for $\beta = 2$ (left), $\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$ (middle), and $\beta = \frac{1}{2} + \frac{1}{\beta^2} \approx 1.325$ (right).

The set of $(-\beta)$ -integers is therefore defined by

$$\mathbb{Z}_{-\beta} = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0).$$

These are the numbers

$$\sum_{k=0}^{n-1} a_k (-\beta)^k \quad \text{such that} \quad \frac{-\beta}{\beta+1} \leq \sum_{k=0}^{m-1} a_k (-\beta)^{k-m} < \frac{1}{\beta+1} \quad \text{for all } 1 \leq m \leq n.$$

Note that, in the case of β -integers, we have to add $-\mathbb{Z}_\beta^+$ to \mathbb{Z}_β^+ in order to obtain a set resembling \mathbb{Z} . In the case of $(-\beta)$ -integers, this is not necessary because the $(-\beta)$ -transformation allows to represent positive and negative numbers.

It is not difficult to see that $\mathbb{Z}_{-\beta} = \mathbb{Z} = \mathbb{Z}_\beta$ when $\beta \in \mathbb{Z}$, $\beta \geq 2$. Some other properties of $\mathbb{Z}_{-\beta}$ can be found in [1], mainly for the case when $T_{-\beta}^n(\frac{-\beta}{\beta+1}) \leq 0$ and $T_{-\beta}^{2n-1}(\frac{-\beta}{\beta+1}) \geq \frac{1-|\beta|}{\beta}$ for all $n \geq 1$. (Note that $T_{-\beta}^n(\frac{-\beta}{\beta+1}) \in (\frac{1}{\beta+1} - \frac{|\beta|}{\beta}, \frac{1-|\beta|}{\beta}) \cup (\frac{-\beta-1}{\beta+1}, 0)$ implies $T_{-\beta}^{n+1}(\frac{-\beta}{\beta+1}) > 0$.)

The set

$$V_\beta = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\}$$

plays a similar role for $(-\beta)$ -expansions as the set $\{T_\beta^n(1^-) \mid n \geq 0\}$ for β -expansions. If V_β is a finite set, then we call $\beta > 1$ an *Yrrap number*. Note that these numbers are called *Ito-Sadahiro numbers* in [18], in reference to [14]. However, as the generalised β -transformations in [13] with $E = (1, \dots, 1)$ are, up to conjugation by the map $x \mapsto \frac{1}{\beta+1} - x$, the same as our $(-\beta)$ -transformations, these numbers were already considered by Góra and perhaps by other authors. Therefore, the neutral but intricate name $(-\beta)$ -numbers was chosen in [17], referring to the original name β -numbers for Parry numbers [19]. The name Yrrap number, used in the present paper, refers to the connection with Parry numbers and to the fact that $T_{-\beta}$ is (locally) orientation-reversing.

For any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we describe the sequence of $(-\beta)$ -integers in terms of a two-sided infinite word on a finite alphabet which is a fixed point of an anti-morphism (Theorem 3). Note that the orientation-reversing property of the map $x \mapsto -\beta x$ imposes the use of an anti-morphism instead of a morphism, and that anti-morphisms were considered in a similar context by Enomoto [8].

For $1 < \beta < \frac{1+\sqrt{5}}{2}$, we have $\mathbb{Z}_{-\beta} = \{0\}$, as already proved in [1]. However, our study still makes sense for these bases $(-\beta)$ because we can also describe the sets

$$S_{-\beta}(x) = \lim_{n \rightarrow \infty} (-\beta)^n T_{-\beta}^{-n}(x) \quad (x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})),$$

where the limit set consists of the numbers lying in all but finitely many sets $(-\beta)^n T_{-\beta}^{-n}(x)$, $n \geq 0$. Taking the limit instead of the union over all $n \geq 0$ implies that every $y \in \mathbb{R}$ lies in exactly one set $S_{-\beta}(x)$, $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, see Lemma 2. Note that $T_{-\beta}^2(\frac{-\beta-1}{\beta+1}) \neq \frac{-\beta}{\beta+1}$ when $\beta \notin \mathbb{Z}$. Other properties of the $(-\beta)$ -transformation for $1 < \beta < \frac{1+\sqrt{5}}{2}$ are exhibited in [17].

2. β -INTEGERS

In this section, we consider the structure of β -integers. The results are not new, but it is useful to state and prove them in order to understand the differences with $(-\beta)$ -integers.

Recall that $\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\}$, and let Δ_β^* be the free monoid generated by Δ_β . Elements of Δ_β^* will be considered as words on the alphabet Δ_β , and the operation is the concatenation of words. The β -substitution is the morphism $\varphi_\beta : \Delta_\beta^* \rightarrow \Delta_\beta^*$, defined by

$$\varphi_\beta(x) = \underbrace{11 \cdots 1}_{[\beta x] - 1 \text{ times}} T_\beta(x^-) \quad (x \in \Delta_\beta).$$

Here, 1 is an element of Δ_β and not the identity element of Δ_β^* (which is the empty word). Recall that, as φ_β is a morphism, we have $\varphi_\beta(uv) = \varphi_\beta(u)\varphi_\beta(v)$ for all $u, v \in \Delta_\beta^*$. Since $\varphi_\beta^{n+1}(1) = \varphi_\beta^n(\varphi_\beta(1))$ and $\varphi_\beta(1)$ starts with 1, $\varphi_\beta^n(1)$ is a prefix of $\varphi_\beta^{n+1}(1)$ for every $n \geq 0$.

Theorem 1. For any $\beta > 1$, the set of non-negative β -integers takes the form

$$\mathbb{Z}_\beta^+ = \{z_k \mid k \geq 0\} \quad \text{with} \quad z_k = \sum_{j=1}^k u_j,$$

where $u_1 u_2 \cdots$ is the infinite word with letters in Δ_β which has $\varphi_\beta^n(1)$ as prefix for all $n \geq 0$. The set of differences between consecutive β -integers is Δ_β .

The main observation for the proof of the theorem is the following. We use the notation $|v| = k$ and $L(v) = \sum_{j=1}^k v_j$ for any $v = v_1 \cdots v_k \in \Delta_\beta^k$, $k \geq 0$.

Lemma 1. For any $n \geq 0$, $1 \leq k \leq |\varphi_\beta^n(1)|$, we have

$$T_\beta^n \left(\left[\frac{z_{k-1}}{\beta^n}, \frac{z_k}{\beta^n} \right) \right) = [0, u_k),$$

and $z_{|\varphi_\beta^n(1)|} = L(\varphi_\beta^n(1)) = \beta^n$.

Proof. For $n = 0$, we have $|\varphi_\beta^0(1)| = 1$, $z_0 = 0$, $z_1 = 1$, $u_1 = 1$, thus the statements are true. Suppose that they hold for n , and consider

$$u_1 u_2 \cdots u_{|\varphi_\beta^{n+1}(1)|} = \varphi_\beta^{n+1}(1) = \varphi_\beta(\varphi_\beta^n(1)) = \varphi_\beta(u_1) \varphi_\beta(u_2) \cdots \varphi_\beta(u_{|\varphi_\beta^n(1)|}).$$

Let $1 \leq k \leq |\varphi_\beta^{n+1}(1)|$, and write $u_1 \cdots u_k = \varphi_\beta(u_1 \cdots u_{j-1}) v_1 \cdots v_i$ with $1 \leq j \leq |\varphi_\beta^n(1)|$, $1 \leq i \leq |\varphi_\beta(u_j)|$, i.e., $v_1 \cdots v_i$ is a non-empty prefix of $\varphi_\beta(u_j)$.

For any $x \in (0, 1]$, we have $T_\beta(x^-) = \beta x - [\beta x] + 1$, hence $L(\varphi_\beta(x)) = \beta x$ for $x \in \Delta_\beta$. This yields that

$$z_k = L(u_1 \cdots u_k) = \beta L(u_1 \cdots u_{j-1}) + L(v_1 \cdots v_i) = \beta z_{j-1} + i - 1 + v_i$$

and $z_{k-1} = \beta z_{j-1} + i - 1$, hence

$$\left[\frac{z_{k-1}}{\beta}, \frac{z_k}{\beta} \right) = \left[z_{j-1} + \frac{i-1}{\beta}, z_{j-1} + \frac{i-1+v_i}{\beta} \right) \subseteq [z_{j-1}, z_{j-1} + u_j) = [z_{j-1}, z_j),$$

$$T_\beta^{n+1} \left(\left[\frac{z_{k-1}}{\beta^{n+1}}, \frac{z_k}{\beta^{n+1}} \right) \right) = T_\beta \left(\left[\frac{i-1}{\beta}, \frac{i-1+v_i}{\beta} \right) \right) = [0, v_i) = [0, u_k).$$

Moreover, we have $L(\varphi_\beta^{n+1}(1)) = \beta L(\varphi_\beta^n(1)) = \beta^{n+1}$, thus the statements hold for $n+1$. \square

Proof of Theorem 1. By Lemma 1, we have $z_{|\varphi_\beta^n(1)|} = \beta^n$ for all $n \geq 0$, thus $[0, 1)$ is split into the intervals $[z_{k-1}/\beta^n, z_k/\beta^n)$, $1 \leq k \leq |\varphi_\beta^n(1)|$. Therefore, Lemma 1 yields that

$$T_\beta^{-n}(0) = \{z_{k-1}/\beta^n \mid 1 \leq k \leq |\varphi_\beta^n(1)|\},$$

hence

$$\bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0) = \{z_k \mid k \geq 0\}.$$

Since $u_k \in \Delta_\beta$ for all $k \geq 1$ and $u_{|\varphi_\beta^n(1)|} = T_\beta^n(1^-)$ for all $n \geq 0$, we have

$$\{z_k - z_{k-1} \mid k \geq 1\} = \{u_k \mid k \geq 1\} = \Delta_\beta. \quad \square$$

For the sets $S_\beta(x)$, Lemma 1 gives the following corollary.

Corollary 1. *For any $x \in [0, 1)$, we have*

$$S_\beta(x) = \{z_k + x \mid k \geq 0, u_{k+1} > x\} \subseteq x + S_\beta(0).$$

In particular, we have $S_\beta(x) - x = S_\beta(y) - y$ for all $x, y \in [0, 1)$ with $(x, y] \cap \Delta_\beta = \emptyset$. From the definition of $S_\beta(x)$ and since $x \in \beta T_\beta^{-1}(x)$, we also get that

$$S_\beta(x) = \bigcup_{y \in T_\beta^{-1}(x)} \beta S_\beta(y) \quad (x \in [0, 1)).$$

This shows that $S_\beta(x)$ is the solution of a graph-directed iterated function system (GIFS) when β is a Parry number, cf. [15, Section 3.2].

3. $(-\beta)$ -INTEGERS

We now turn to the discussion of $(-\beta)$ -integers and sets $S_{-\beta}(x)$, $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$.

Lemma 2. *For any $\beta > 1$, $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, we have*

$$S_{-\beta}(x) = \bigcup_{n \geq 0} (-\beta)^n (T_{-\beta}^{-n}(x) \setminus \{\frac{-\beta}{\beta+1}\}) = \bigcup_{y \in T_{-\beta}^{-1}(x)} (-\beta) S_{-\beta}(y).$$

For any $y \in \mathbb{R}$, there exists a unique $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ such that $y \in S_{-\beta}(x)$.

If $T_{-\beta}(x) = x$, then $S_{-\beta}(x) = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(x)$, in particular $S_{-\beta}(0) = \mathbb{Z}_{-\beta}$.

Proof. If $y \in S_{-\beta}(x)$, then we have $\frac{y}{(-\beta)^n} \in T_{-\beta}^{-n}(x)$ for all sufficiently large n , thus $y \in (-\beta)^n (T_{-\beta}^{-n}(x) \setminus \{\frac{-\beta}{\beta+1}\})$ for some $n \geq 0$. On the other hand, $y \in (-\beta)^n (T_{-\beta}^{-n}(x) \setminus \{\frac{-\beta}{\beta+1}\})$ for some $n \geq 0$ implies that $T_{-\beta}^m(\frac{y}{(-\beta)^m}) = T_{-\beta}^n(\frac{y}{(-\beta)^n}) = x$ for all $m \geq n$, thus $y \in S_{-\beta}(x)$. This shows the first equation. Since $x \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ implies that $x \in (-\beta) (T_{-\beta}^{-1}(x) \setminus \{\frac{-\beta}{\beta+1}\})$, we obtain that $S_{-\beta}(x) = \bigcup_{y \in T_{-\beta}^{-1}(x)} (-\beta) S_{-\beta}(y)$ for all $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$.

For any $y \in \mathbb{R}$, we have $y \in S_{-\beta}(T_{-\beta}^{-n}(\frac{y}{(-\beta)^n}))$ for all $n \geq 0$ such that $\frac{y}{(-\beta)^n} \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, thus $y \in S_{-\beta}(x)$ for some $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$. To show that this x is unique, let $y \in S_{-\beta}(x)$ and $y \in S_{-\beta}(x')$ for some $x, x' \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$. Then we have $y \in (-\beta)^n (T_{-\beta}^{-n}(x) \setminus \{\frac{-\beta}{\beta+1}\})$ and $y \in (-\beta)^m (T_{-\beta}^{-m}(x') \setminus \{\frac{-\beta}{\beta+1}\})$ for some $m, n \geq 0$, thus $x = T_{-\beta}^n(\frac{y}{(-\beta)^n}) = T_{-\beta}^m(\frac{y}{(-\beta)^m}) = x'$.

If $T_{-\beta}^n(\frac{-\beta}{\beta+1}) = x = T_{-\beta}(x)$, then $T_{-\beta}^{n+2}(\frac{-\beta^{-1}}{\beta+1}) = T_{-\beta}^{n+1}(\frac{-\beta}{\beta+1}) = T_{-\beta}(x) = x$ yields that $(-\beta)^n \frac{-\beta}{\beta+1} \in S_{-\beta}(x)$, which shows that $S_{-\beta}(x) = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(x)$ when $T_{-\beta}(x) = x$. \square

The first two statements of the following proposition can also be found in [1].

Proposition 1. *For any $\beta > 1$, we have $(-\beta) \mathbb{Z}_{-\beta} \subseteq \mathbb{Z}_{-\beta}$.*

If $\beta < (1 + \sqrt{5})/2$, then $\mathbb{Z}_{-\beta} = \{0\}$.

If $\beta \geq (1 + \sqrt{5})/2$, then

$$\mathbb{Z}_{-\beta} \cap (-\beta)^n [-\beta, 1] = \{(-\beta)^n, (-\beta)^{n+1}\} \cup (-\beta)^{n+2} (T_{-\beta}^{-n-2}(0) \cap (\frac{-1}{\beta}, \frac{1}{\beta^2}))$$

for all $n \geq 0$, in particular

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \begin{cases} \{-\beta, -\beta + 1, \dots, -\beta + \lfloor \beta \rfloor, 0, 1\} & \text{if } \beta^2 \geq \lfloor \beta \rfloor(\beta + 1), \\ \{-\beta, -\beta + 1, \dots, -\beta + \lfloor \beta \rfloor - 1, 0, 1\} & \text{if } \beta^2 < \lfloor \beta \rfloor(\beta + 1). \end{cases}$$

Proof. The inclusion $(-\beta)\mathbb{Z}_{-\beta} \subseteq \mathbb{Z}_{-\beta}$ is a consequence of Lemma 2 and $0 \in T_{-\beta}^{-1}(0)$.

If $\beta < \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} < \frac{-\beta}{\beta+1}$, hence $T_{-\beta}^{-1}(0) = \{0\}$ and $\mathbb{Z}_{-\beta} = \{0\}$, see Figure 1 (right).

If $\beta \geq \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} \in T_{-\beta}^{-1}(0)$ implies $1 \in \mathbb{Z}_{-\beta}$, thus $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$. Clearly,

$$(-\beta)^{n+2} (T_{-\beta}^{-n-2}(0) \cap (\frac{-1}{\beta}, \frac{1}{\beta^2})) \subseteq \mathbb{Z}_{-\beta} \cap (-\beta)^n (-\beta, 1).$$

To show the other inclusion, let $z \in (-\beta)^m T_{-\beta}^{-m}(0) \cap (-\beta)^n (-\beta, 1)$ for some $m \geq 0$. If $z \neq (-\beta)^m \frac{-\beta}{\beta+1}$, then $\frac{z}{(-\beta)^m} \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ and $\frac{z}{(-\beta)^{n+2}} \in (\frac{-1}{\beta}, \frac{1}{\beta^2}) \subseteq (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ imply that $T_{-\beta}^{n+2}(\frac{z}{(-\beta)^{n+2}}) = T_{-\beta}^m(\frac{z}{(-\beta)^m}) = 0$. If $z = (-\beta)^m \frac{-\beta}{\beta+1}$, then

$$T_{-\beta}^{n+2}(\frac{z}{(-\beta)^{n+2}}) = T_{-\beta}^{n+2}(\frac{(-\beta)^{m-n-1}}{\beta+1}) = T_{-\beta}^{m+2}(\frac{-\beta^{-1}}{\beta+1}) = T_{-\beta}^{m+1}(\frac{-\beta}{\beta+1}) = T_{-\beta}(0) = 0,$$

where we have used that $\frac{z}{(-\beta)^{n+2}} \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$ implies $m \leq n$. Therefore, we have $z \in (-\beta)^{n+2} T_{-\beta}^{-n-2}(0)$ for all $z \in \mathbb{Z}_{-\beta} \cap (-\beta)^n (-\beta, 1)$.

Consider now $n = 0$, then

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \{-\beta, 1\} \cup \{z \in (-\beta, 1) \mid T_{-\beta}^2(z/\beta^2) = 0\}.$$

Since $\frac{-\lfloor \beta \rfloor}{\beta} \geq \frac{-\beta}{\beta+1}$ if and only if $\beta^2 \geq \lfloor \beta \rfloor(\beta + 1)$, we obtain that

$$(-\beta) T_{-\beta}^{-1}(0) = \begin{cases} \{0, 1, \dots, \lfloor \beta \rfloor\} & \text{if } \beta^2 \geq \lfloor \beta \rfloor(\beta + 1), \\ \{0, 1, \dots, \lfloor \beta \rfloor - 1\} & \text{if } \beta^2 < \lfloor \beta \rfloor(\beta + 1). \end{cases}$$

If $T_{-\beta}^2(z/\beta^2) = 0$, then $z = -a_1\beta + a_0$ with $a_0 \in (-\beta) T_{-\beta}^{-1}(0)$, $a_1 \in \{0, 1, \dots, \lfloor \beta \rfloor\}$, and $\mathbb{Z}_{-\beta} \cap [-\beta, 1]$ consists of those numbers $-a_1\beta + a_0$ lying in $[-\beta, 1]$. \square

Proposition 1 shows that the maximal difference between consecutive $(-\beta)$ -integers exceeds 1 whenever $\beta^2 < \lfloor \beta \rfloor(\beta + 1)$, i.e., $T_{-\beta}(\frac{-\beta}{\beta+1}) < 0$. For an example, this was also proved in [1]. In Examples 3 and 4, we see that the distance between two consecutive $(-\beta)$ -integers can be even greater than 2, and the structure of $\mathbb{Z}_{-\beta}$ can be quite complicated. Therefore, we adapt a slightly different strategy as for \mathbb{Z}_β .

In the following, we always assume that the set

$$V'_\beta = V_\beta \cup \{0\} = \{T_{-\beta}^n(\frac{-\beta}{\beta+1}) \mid n \geq 0\} \cup \{0\}$$

is finite, i.e., β is an Yrrap number, and let β be fixed. For $x \in V'_\beta$, let

$$r_x = \min \{y \in V'_\beta \cup \{\frac{1}{\beta+1}\} \mid y > x\}, \quad \hat{x} = \frac{x+r_x}{2}, \quad J_x = \{x\} \quad \text{and} \quad J_{\hat{x}} = (x, r_x).$$

Then $\{J_a \mid a \in A_\beta\}$ forms a partition of $[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, where

$$A_\beta = V'_\beta \cup \hat{V}'_\beta, \quad \text{with} \quad \hat{V}'_\beta = \{\hat{x} \mid x \in V'_\beta\}.$$

We have $T_{-\beta}(J_x) = J_{T_{-\beta}(x)}$ for every $x \in V'_\beta$, and the following lemma shows that the image of every $J_{\hat{x}}$, $x \in V'_\beta$, is a union of intervals J_a , $a \in A_\beta$.

Lemma 3. *Let $x \in V'_\beta$ and write*

$$J_{\hat{x}} \cap T_{-\beta}^{-1}(V'_\beta) = \{y_1, \dots, y_m\}, \quad \text{with } x = y_0 < y_1 < \dots < y_m < y_{m+1} = r_x.$$

For any $0 \leq i \leq m$, we have

$$T_{-\beta}((y_i, y_{i+1})) = J_{\hat{x}_i} \quad \text{with } x_i = \lim_{y \rightarrow y_{i+1}^-} T_{-\beta}(y), \quad \text{i.e., } \hat{x}_i = T_{-\beta}\left(\frac{y_i + y_{i+1}}{2}\right),$$

and $\beta(y_{i+1} - y_i) = \lambda(J_{\hat{x}_i})$, where λ denotes the Lebesgue measure.

Proof. Since $T_{-\beta}$ maps no point in (y_i, y_{i+1}) to $\frac{-\beta}{\beta+1} \in V'_\beta$, the map is continuous on this interval and $\lambda(T_{-\beta}((y_i, y_{i+1}))) = \beta(y_{i+1} - y_i)$. We have $x_i \in V'_\beta$ since $x_i = T_{-\beta}(y_{i+1})$ in case $y_{i+1} < \frac{1}{\beta+1}$, and $x_i = \frac{-\beta}{\beta+1}$ in case $y_{i+1} = \frac{1}{\beta+1}$. Since $y_i = \max\{y \in T_{-\beta}^{-1}(V'_\beta) \mid y < y_{i+1}\}$, we obtain that $r_{x_i} = \lim_{y \rightarrow y_i^+} T_{-\beta}(y)$, thus $T_{-\beta}((y_i, y_{i+1})) = (x_i, r_{x_i})$. \square

In view of Lemma 3, we define an anti-morphism $\psi_\beta : A_\beta^* \rightarrow A_\beta^*$ by

$$\psi_\beta(x) = T_{-\beta}(x) \quad \text{and} \quad \psi_\beta(\hat{x}) = \hat{x}_m T_{-\beta}(y_m) \cdots \hat{x}_1 T_{-\beta}(y_1) \hat{x}_0 \quad (x \in V'_\beta),$$

with m , x_i and y_i as in Lemma 3. Here, anti-morphism means that $\psi_\beta(uv) = \psi_\beta(v)\psi_\beta(u)$ for all $u, v \in A_\beta^*$. Now, the last letter of $\psi_\beta(\hat{0})$ is \hat{t} , with $t = \max\{x \in V_\beta \mid x < 0\}$, and the first letter of $\psi_\beta(\hat{t})$ is $\hat{0}$. Therefore, $\psi_\beta^{2n}(\hat{0})$ is a prefix of $\psi_\beta^{2n+2}(\hat{0}) = \psi_\beta^{2n}(\psi_\beta^2(\hat{0}))$ and $\psi_\beta^{2n+1}(\hat{0})$ is a suffix of $\psi_\beta^{2n+3}(\hat{0})$ for every $n \geq 0$.

Theorem 2. *For any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we have*

$$\mathbb{Z}_{-\beta} = \{z_k \mid k \in \mathbb{Z}, u_{2k} = 0\} \quad \text{with} \quad z_k = \begin{cases} \sum_{j=1}^k \lambda(J_{u_{2j-1}}) & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} \lambda(J_{u_{-2j+1}}) & \text{if } k \leq 0, \end{cases}$$

where $\cdots u_{-1}u_0u_1\cdots$ is the two-sided infinite word on the finite alphabet A_β such that $u_0 = 0$, $\psi_\beta^{2n}(\hat{0})$ is a prefix of $u_1u_2\cdots$ and $\psi_\beta^{2n+1}(\hat{0})$ is a suffix of $\cdots u_{-2}u_{-1}$ for all $n \geq 0$.

Note that $\cdots u_{-1}u_0u_1\cdots$ is a fixed point of ψ_β , with u_0 being mapped to u_0 .

The following lemma is the analogue of Lemma 1. We use the notation

$$L(v) = \sum_{j=1}^k \lambda(J_{v_j}) \quad \text{if } v = v_1 \cdots v_k \in A_\beta^k.$$

Note that $u_{2k} \in V'_\beta$ and $u_{2k+1} \in \widehat{V}'_\beta$ for all $k \in \mathbb{Z}$, thus $\lambda(J_{u_{2k}}) = 0$ for all $k \in \mathbb{Z}$.

Lemma 4. *For any $n \geq 0$, $0 \leq k < |\psi_\beta^n(\hat{0})|/2$, we have*

$$T_{-\beta}^n\left(\frac{z_{(-1)^{nk}}}{(-\beta)^n}\right) = u_{(-1)^{n2k}}, \quad T_{-\beta}^n\left(\left(\frac{z_{(-1)^{nk}}}{(-\beta)^n}, \frac{z_{(-1)^{n(k+1)}}}{(-\beta)^n}\right)\right) = J_{u_{(-1)^{n(2k+1)}},$$

and $z_{(-1)^n(|\psi_\beta^n(\hat{0})|+1)/2} = (-1)^n L(\psi_\beta^n(\hat{0})) = \lambda(J_{\hat{0}}) (-\beta)^n = r_0 (-\beta)^n$.

Proof. The statements are true for $n = 0$ since $|\psi_\beta^0(\widehat{0})| = 1$, $z_0 = 0$, $z_1 = \lambda(J_\widehat{0}) = r_0$.

Suppose that they hold for even n , and consider

$$u_{-|\psi_\beta^{n+1}(\widehat{0})|} \cdots u_{-2} u_{-1} = \psi_\beta^{n+1}(\widehat{0}) = \psi_\beta(\psi_\beta^n(\widehat{0})) = \psi_\beta(u_{|\psi_\beta^n(\widehat{0})|}) \cdots \psi_\beta(u_2) \psi_\beta(u_1).$$

Let $0 \leq k < |\psi_\beta^{n+1}(\widehat{0})|/2$, and write

$$u_{-2k-1} \cdots u_{-1} = v_{-2i-1} \cdots v_{-1} \psi_\beta(u_1 \cdots u_{2j})$$

with $0 \leq j < |\psi_\beta^n(\widehat{0})|/2$, $0 \leq i < |\psi_\beta(u_{2j+1})|/2$, i.e., $u_{-2i-1} \cdots u_{-1}$ is a suffix of $\psi_\beta(u_{2j+1})$.

By Lemma 3, we have $L(\psi_\beta(\widehat{x})) = \beta \lambda(J_{\widehat{x}})$ for any $x \in V'_\beta$. This yields that

$$-z_{-k-1} = \beta L(u_1 \cdots u_{2j}) + L(v_{-2i-1} \cdots v_{-1}) = \beta z_j + L(v_{-2i-1} \cdots v_{-1})$$

and $-z_{-k} = \beta z_j + L(v_{-2i} \cdots v_{-1})$. By the induction hypothesis, we obtain that

$$\begin{aligned} T_{-\beta}^{n+1} \left(\frac{z_{-k}}{(-\beta)^{n+1}} \right) &= T_{-\beta}^{n+1} \left(\frac{z_j}{(-\beta)^n} - \frac{L(v_{-2i} \cdots v_{-1})}{(-\beta)^{n+1}} \right) \\ &= \begin{cases} T_{-\beta}(u_{2j}) = \psi_\beta(u_{2j}) = u_{-2k} & \text{if } i = 0, \\ T_{-\beta}(x + L(v_{-2i} \cdots v_{-1})/\beta) = T_{-\beta}(y_i) = v_{-2i} = u_{-2k} & \text{if } i \geq 1, \end{cases} \end{aligned}$$

where the y_i 's are the numbers from Lemma 3 for $\widehat{x} = u_{2j+1}$, and

$$T_{-\beta}^{n+1} \left(\left(\frac{z_{-k}}{(-\beta)^{n+1}}, \frac{z_{-k-1}}{(-\beta)^{n+1}} \right) \right) = T_{-\beta}((y_i, y_{i+1})) = J_{v_{-2i-1}} = J_{u_{-2k-1}}.$$

Moreover, we have $L(\psi_\beta^{n+1}(\widehat{0})) = \beta L(\psi_\beta^n(\widehat{0})) = r_0 \beta^{n+1}$, thus the statements hold for $n + 1$.

The proof for odd n runs along the same lines and is therefore omitted. \square

Proof of Theorem 2. By Lemma 4, we have $z_{(-1)^n(|\psi_\beta^n(\widehat{0})|+1)/2} = r_0 (-\beta)^n$ for all $n \geq 0$, thus $[0, r_0)$ splits into the intervals $(z_{(-1)^n k} (-\beta)^{-n}, z_{(-1)^n (k+1)} (-\beta)^{-n})$ and points $z_{(-1)^n k} (-\beta)^{-n}$, $0 \leq k < |\psi_\beta^n(\widehat{0})|/2$, hence

$$T_{-\beta}^{-n}(0) \cap [0, r_0) = \{z_{(-1)^n k} (-\beta)^{-n} \mid 0 \leq k < |\psi_\beta^n(\widehat{0})|/2, u_{(-1)^n 2k} = 0\}.$$

Let $m \geq 1$ be such that $\beta^{2m} r_0 \geq \frac{1}{\beta+1}$. Then we have $(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}) \subseteq (-\beta^{2m+1} r_0, \beta^{2m} r_0)$, and

$$T_{-\beta}^{-n}(0) \setminus \left\{ \frac{-\beta}{\beta+1} \right\} \subseteq (-\beta)^{2m} (T_{-\beta}^{-n-2m}(0) \cap [0, r_0)) \cup (-\beta)^{2m+1} (T_{-\beta}^{-n-2m-1}(0) \cap [0, r_0)),$$

thus

$$\bigcup_{n \geq 0} (-\beta)^n (T_{-\beta}^{-n}(0) \setminus \left\{ \frac{-\beta}{\beta+1} \right\}) = \bigcup_{n \geq 0} (-\beta)^n (T_{-\beta}^{-n}(0) \cap [0, r_0)) = \{z_k \mid k \in \mathbb{Z}, u_{2k} = 0\}.$$

Together with Lemma 2, this proves the theorem. \square

As in the case of positive bases, the word $\cdots u_{-1} u_0 u_1 \cdots$ also describes the sets $S_{-\beta}(x)$. Theorem 2 and Lemma 4 give the following corollary.

Corollary 2. *For any $x \in V'_\beta$, $y \in J_{\widehat{x}}$, we have*

$$S_{-\beta}(x) = \{z_k \mid k \in \mathbb{Z}, u_{2k} = x\} \quad \text{and} \quad S_{-\beta}(y) = \{z_k + y - x \mid k \in \mathbb{Z}, u_{2k+1} = \widehat{x}\}.$$

Lemma 2 and Corollary 2 imply that $S_{-\beta}(x)$ is the solution of a GIFS for any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, $x \in [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, cf. the end of Section 2.

Recall that our main goal is to understand the structure of $\mathbb{Z}_{-\beta}$ (in case $\beta \geq (1 + \sqrt{5})/2$), i.e., to describe the occurrences of 0 in the word $\cdots u_{-1}u_0u_1 \cdots$ defined in Theorem 2 and the words between two successive occurrences. Let

$$R_\beta = \{u_{2k}u_{2k+1} \cdots u_{2s(k)-1} \mid k \in \mathbb{Z}, u_{2k} = 0\} \quad \text{with} \quad s(k) = \min\{j \in \mathbb{Z} \mid u_{2j} = 0, j > k\}$$

be the set of return words of 0 in $\cdots u_{-1}u_0u_1 \cdots$. Note that $s(k)$ is defined for all $k \in \mathbb{Z}$ since $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$ by Proposition 1.

For any $w \in R_\beta$, the word $\psi_\beta(w_0)$ is a factor of $\cdots u_{-1}u_0u_1 \cdots$ starting and ending with 0, thus we can write $\psi_\beta(w_0) = w_1 \cdots w_m 0$ with $w_j \in R_\beta$, $1 \leq j \leq m$, and set

$$\varphi_{-\beta}(w) = w_1 \cdots w_m.$$

This defines an anti-morphism $\varphi_{-\beta} : R_\beta^* \rightarrow R_\beta^*$, which plays the role of the β -substitution.

Since $\cdots u_{-1}u_0u_1 \cdots$ is generated by $u_1 = \widehat{0}$, as described in Theorem 2, we consider $w_\beta = u_0u_1 \cdots u_{2s(0)-1}$. We have

$$[0, 1] = [0, \frac{1}{\beta+1}) \cup [\frac{1}{\beta+1}, 1], \quad T_{-\beta}((-\beta)^{-1}[\frac{1}{\beta+1}, 1]) = [\frac{-\beta}{\beta+1}, 0],$$

thus $L(w_\beta) = 1$ and

$$w_\beta = 0 \widehat{0} x_1 \widehat{x}_1 \cdots x_m \widehat{x}_m x_{-\ell} \widehat{x}_{-\ell} \cdots x_{-1} \widehat{x}_{-1},$$

where the x_j are defined by $V'_\beta = \{x_{-\ell}, \dots, x_{-1}, 0, x_1, \dots, x_m\}$, $x_{-\ell} < \cdots < x_{-1} < 0 < x_1 < \cdots < x_m$.

Theorem 3. *For any Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we have*

$$\mathbb{Z}_{-\beta} = \{z'_k \mid k \in \mathbb{Z}\} \quad \text{with} \quad z'_k = \begin{cases} \sum_{j=1}^k L(u'_j) & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} L(u'_{-j}) & \text{if } k \leq 0, \end{cases}$$

where $\cdots u'_{-2}u'_{-1}u'_1u'_2 \cdots$ is the two-sided infinite word on the finite alphabet R_β such that $\varphi_{-\beta}^{2n}(w_\beta)$ is a prefix of $u'_1u'_2 \cdots$ and $\varphi_{-\beta}^{2n+1}(w_\beta)$ is a suffix of $\cdots u'_{-2}u'_{-1}$ for all $n \geq 0$.

The set of distances between consecutive $(-\beta)$ -integers is

$$\Delta_{-\beta} = \{z'_{k+1} - z'_k \mid k \in \mathbb{Z}\} = \{L(w) \mid w \in R_\beta\}.$$

Note that the index 0 is omitted in $\cdots u'_{-2}u'_{-1}u'_1u'_2 \cdots$ for reasons of symmetry.

Proof. The definitions of $\cdots u_{-1}u_0u_1 \cdots$ in Theorem 2 and of $\varphi_{-\beta}$ imply that $\cdots u'_{-2}u'_{-1}u'_1u'_2 \cdots$ is the derived word of $\cdots u_{-1}u_0u_1 \cdots$ with respect to R_β , that is

$$u'_k = u_{|u'_1 \cdots u'_{k-1}|} \cdots u_{|u'_1 \cdots u'_k|-1}, \quad u'_{-k} = u_{-|u'_{-k} \cdots u'_{-1}|} \cdots u_{-|u'_{-k} \cdots u'_{-1}|-1} \quad (k \geq 1)$$

with

$$\{|u'_1 \cdots u'_{k-1}| \mid k \geq 1\} \cup \{-|u'_{-k} \cdots u'_{-1}| \mid k \geq 1\} = \{k \in \mathbb{Z} \mid u_k = 0\}.$$

where $\hat{0}$ marks the central letter u_0 . The return word of $\hat{0}$ starting at u_0 is $w_\beta = 0\hat{0}t\hat{t}$. The image $\psi_\beta(w_\beta 0) = 0\hat{0}t\hat{t}0\hat{t}0$ contains the return words w_β and $0\hat{t}$. Since $\psi_\beta(0\hat{t}0) = 0\hat{0}t\hat{t}0$, there are no other return words of $\hat{0}$, i.e., $R_\beta = \{A, B\}$ with $A = 0\hat{0}t\hat{t}$, $B = 0\hat{t}$. Therefore, $\dots u'_{-2}u'_{-1}u'_1u'_2 \dots$ is a two-sided fixed point of the anti-morphism

$$\varphi_{-\beta} : A \mapsto AB, \quad B \mapsto A,$$

with

$$u'_{-13} \dots u'_{-1} u'_1 \dots u'_{21} = AABAABABAABAB AABAABABAABABAABAB.$$

We have $\lambda(J_{\hat{0}}) = \frac{1}{\beta^2}$, $\lambda(J_{\hat{t}}) = \frac{1}{\beta}$, thus $L(A) = 1$, $L(B) = \frac{1}{\beta} = \beta - 1$, and some $(-\beta)$ -integers are shown in Figure 2. Note that $(-\beta)^n$ can also be represented as $(-\beta)^{n+2} + (-\beta)^{n+1}$.

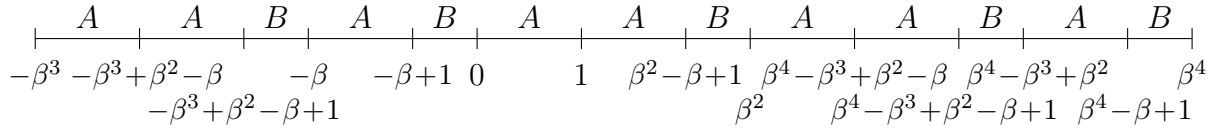


FIGURE 2. The $(-\beta)$ -integers in $[-\beta^3, \beta^4]$, $\beta = (1 + \sqrt{5})/2$.

Example 2. Let $\beta = \frac{3+\sqrt{5}}{2}$, i.e., $\beta^2 = 3\beta - 1$, then the $(-\beta)$ -transformation is depicted in Figure 3, where $t_0 = \frac{-\beta}{\beta+1}$, $t_1 = T_{-\beta}(t_0) = \frac{\beta^2}{\beta+1} - 2 = \frac{-\beta-1}{\beta+1}$, $T_{-\beta}(t_1) = \frac{1}{\beta+1} - 1 = t_0$. Therefore, $V'_\beta = \{t_0, t_1, 0\}$ and the anti-morphism $\psi_\beta : A_\beta^* \rightarrow A_\beta^*$ is defined by

$$\psi_\beta : t_0 \mapsto t_1, \quad \hat{t}_0 \mapsto \hat{t}_0 \hat{t}_1 \hat{t}_1 \hat{0} \hat{0} t_0 \hat{t}_0, \quad t_1 \mapsto t_0, \quad \hat{t}_1 \mapsto \hat{0}, \quad 0 \mapsto 0, \quad \hat{0} \mapsto \hat{t}_0 \hat{t}_1 \hat{t}_1,$$

which has the two-sided fixed point

$$\dots \underbrace{0}_{\psi_\beta(0)} \underbrace{\hat{0}}_{\psi_\beta(\hat{t}_1)} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\hat{t}_0 \hat{t}_1 \hat{t}_1 \hat{0} \hat{0} t_0 \hat{t}_0}_{\psi_\beta(\hat{t}_0)} \underbrace{t_1}_{\psi_\beta(t_0)} \underbrace{\hat{t}_0 \hat{t}_1 \hat{t}_1}_{\psi_\beta(\hat{0})} \underbrace{\hat{0}}_{\psi_\beta(0)} \underbrace{\hat{0}}_{\psi_\beta(\hat{t}_1)} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\hat{t}_0 \hat{t}_1 \hat{t}_1 \hat{0} \hat{0} t_0 \hat{t}_0}_{\psi_\beta(\hat{t}_0)} \dots,$$

where $\hat{0}$ marks the central letter u_0 . We have $w_\beta = 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1$ and

$$\begin{aligned} \psi_\beta : \quad & 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0 \mapsto 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0, \\ & 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0 \mapsto 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0, \\ & 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0 \mapsto 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1 0. \end{aligned}$$

Note that $0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1$ and $0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1$ differ only by a letter in V'_β , and correspond therefore to intervals of the same length. Since the letters t_0 and t_1 are never mapped to 0 , we identify these two return words. This means that $R_\beta = \{A, B\}$ with $A = 0\hat{0}t_0\hat{t}_0\hat{t}_1\hat{t}_1$, $B = 0\hat{0}t_0\hat{t}_0\{t_0, t_1\}\hat{t}_0\hat{t}_1\hat{t}_1$, and

$$\dots u'_{-2}u'_{-1} u'_1u'_2 \dots = \dots ABBABABBAB ABBABABBAB \dots$$

is a two-sided fixed point of the anti-morphism

$$\varphi_{-\beta} : A \mapsto AB, \quad B \mapsto ABB.$$

We have $L(A) = 1$, $L(B) = \beta - 1 > 1$, and some $(-\beta)$ -integers are shown in Figure 3.

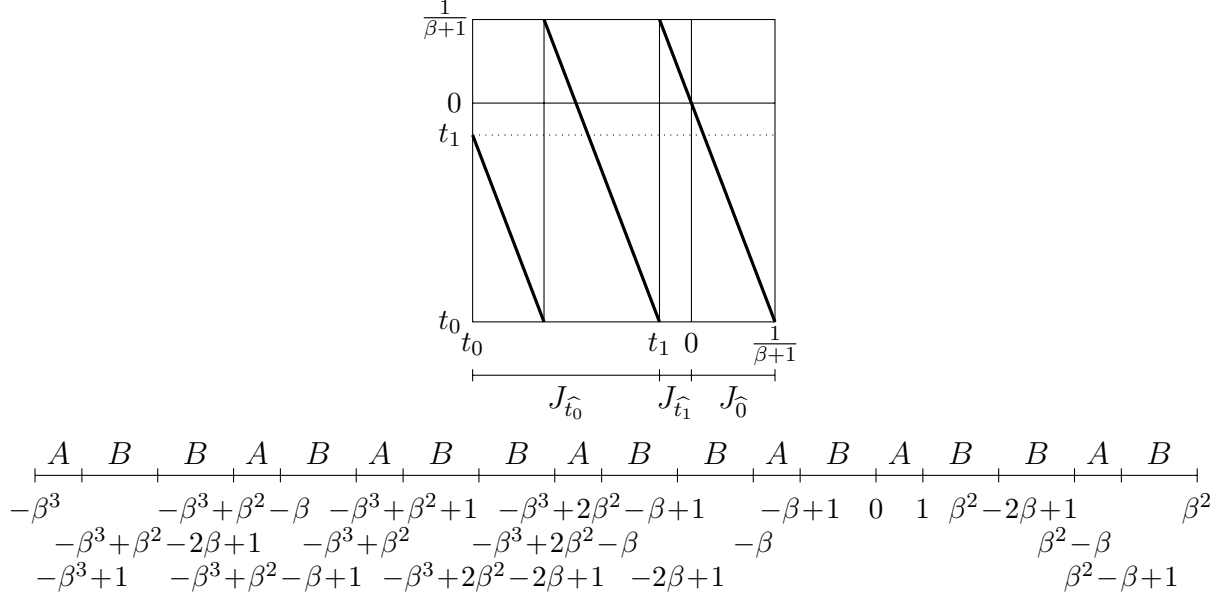


FIGURE 3. The $(-\beta)$ -transformation and $\mathbb{Z}_{-\beta} \cap [-\beta^3, \beta^2]$, $\beta = (3 + \sqrt{5})/2$.

We remark that it is sufficient to consider the elements of \widehat{V}_β when one is only interested in $\mathbb{Z}_{-\beta}$. This is made precise in the following proposition.

Proposition 2. *Let β and $\cdots u_{-1}u_0u_1\cdots$ be as in Theorem 2, $t = \max\{x \in V_\beta \mid x < 0\}$.*

For any $k \in \mathbb{Z}$, $u_{2k} = 0$ is equivalent to $u_{2k-1} = \widehat{t}$ or $u_{2k+1} = \widehat{0}$.

If $0 \notin V_\beta$ or the size of V_β is even, then $u_{2k} = 0$ is equivalent to $u_{2k-1} = \widehat{t}$.

If $0 \notin V_\beta$ or the size of V_β is odd, then $u_{2k} = 0$ is equivalent to $u_{2k+1} = \widehat{0}$.

Proof. Let $k \in \mathbb{Z}$ and $m \geq 0$ such that $z_k/\beta^{2m} \in (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$. Then we have

- $u_{2k} = 0$ if and only if $T_{-\beta}^{2m}(z_k/\beta^{2m}) = 0$,
- $u_{2k-1} = \widehat{t}$ if and only if $\lim_{y \rightarrow z_k^-} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$,
- $u_{2k+1} = \widehat{0}$ if and only if $\lim_{y \rightarrow z_k^+} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$.

Thus $u_{2k} = 0$, $u_{2k-1} = \widehat{t}$ and $u_{2k+1} = \widehat{0}$ are equivalent when $T_{-\beta}^{2m}$ is continuous at z_k/β^{2m} .

Assume from now on that z_k/β^{2m} is a discontinuity point of $T_{-\beta}^{2m}$. Then $T_{-\beta}^\ell(z_k/\beta^{2m}) = \frac{-\beta}{\beta+1}$ for some $1 \leq \ell \leq 2m$ and, if ℓ is minimal with this property,

$$\lim_{y \rightarrow z_k^-} T_{-\beta}^{2\lceil \ell/2 \rceil + 1}(y/\beta^{2m}) = \frac{-\beta}{\beta+1} \quad \text{and} \quad \lim_{y \rightarrow z_k^+} T_{-\beta}^{2\lceil \ell/2 \rceil}(y/\beta^{2m}) = \frac{-\beta}{\beta+1}.$$

Hence, if $0 \notin V_\beta$, we cannot have $u_{2k} = 0$, $u_{2k-1} = \widehat{t}$ or $u_{2k+1} = \widehat{0}$ at a discontinuity point, which proves the proposition in this case. If $0 \in V_\beta$, then $T_{-\beta}^{\#V_\beta-1}(\frac{-\beta}{\beta+1}) = 0$, thus

- $T_{-\beta}^j(z_k/\beta^{2m}) = 0$ if and only if $j \geq \ell + \#V_\beta - 1$,
- $\lim_{y \rightarrow z_k^-} T_{-\beta}^j(y/\beta^{2m}) = 0$ if and only if $j \geq 2\lfloor \ell/2 \rfloor + \#V_\beta$,
- $\lim_{y \rightarrow z_k^+} T_{-\beta}^j(y/\beta^{2m}) = 0$ if and only if $j \geq 2\lceil \ell/2 \rceil + \#V_\beta - 1$.

Since $2\lfloor \ell/2 \rfloor \geq \ell - 1$ and $2\lceil \ell/2 \rceil \geq \ell$, we obtain $u_{2k} = 0$ whenever $u_{2k-1} = \hat{t}$ or $u_{2k+1} = \hat{0}$. If $\#V_\beta$ is even, then $u_{2k} = 0$ implies that $u_{2k-1} = \hat{t}$ since $2m \geq \ell + \#V_\beta - 1$ implies that $2m \geq 2\lfloor \ell/2 \rfloor + \#V_\beta$. If $\#V_\beta$ is odd, then $u_{2k} = 0$ implies that $u_{2k+1} = \hat{0}$ since $2m \geq \ell + \#V_\beta - 1$ implies that $2m \geq 2\lceil \ell/2 \rceil + \#V_\beta - 1$. This proves the proposition. \square

By Proposition 2, it suffices to consider the anti-morphism $\hat{\psi}_\beta : \hat{V}'_\beta \rightarrow \hat{V}'_\beta$ defined by

$$\hat{\psi}_\beta(\hat{x}) = \hat{x}_m \cdots \hat{x}_1 \hat{x}_0 \quad \text{when} \quad \psi_\beta(\hat{x}) = \hat{x}_m T_{-\beta}(y_m) \cdots \hat{x}_1 T_{-\beta}(y_1) \hat{x}_0 \quad (x \in V'_\beta).$$

Then $\Delta_{-\beta}$ is given by the set \hat{R}_β which consists of the return words of $\hat{0}$ when $0 \notin V_\beta$ or the size of V_β is odd. When $0 \in V_\beta$ and the size of V_β is even, as in Example 1, then \hat{R}_β consists of the words $w\hat{t}$ such that $\hat{t}w$ is a return word of \hat{t} .

Example 3. Let $\beta > 1$ with $\beta^3 = 2\beta^2 + 1$, i.e., $\beta \approx 2.206$, and let $t_n = T_{-\beta}^n(\frac{-\beta}{\beta+1})$ for $n \geq 0$. Then we have

$$t_0 = \frac{-\beta}{\beta+1}, \quad t_1 = \frac{\beta^2}{\beta+1} - 2 = \frac{\beta^{-1}-2}{\beta+1}, \quad t_2 = \frac{2\beta-1}{\beta+1} - 1 = \frac{\beta^{-2}}{\beta+1}, \quad t_3 = \frac{-\beta^{-1}}{\beta+1}, \quad t_4 = \frac{1}{\beta+1} - 1 = t_0,$$

see Figure 4. The anti-morphism $\hat{\psi}_\beta : \hat{V}'_\beta \rightarrow \hat{V}'_\beta$ is therefore defined by

$$\hat{\psi}_\beta : \hat{t}_0 \mapsto \hat{t}_2 \hat{t}_0, \quad \hat{t}_1 \mapsto \hat{t}_0 \hat{t}_1 \hat{t}_3 \hat{0}, \quad \hat{t}_3 \mapsto \hat{0} \hat{t}_2, \quad \hat{0} \mapsto \hat{t}_3, \quad \hat{t}_2 \mapsto \hat{t}_0 \hat{t}_1.$$

Since $0 \notin V_\beta$, we consider return words of $\hat{0}$ in the $\hat{\psi}_\beta$ -images of $\hat{w}_\beta = \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3$:

$$\begin{aligned} \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3 &\mapsto \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3\hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3, \\ \hat{0}\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3 &\mapsto \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3\hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3, \\ \hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3 &\mapsto \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3\hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3, \\ \hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3 &\mapsto \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3\hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_0\hat{t}_1\hat{t}_3\hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3, \\ \hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_0\hat{t}_1\hat{t}_3 &\mapsto \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3\hat{0}\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3\hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3. \end{aligned}$$

Hence $\hat{R}_\beta = \{A, B, C, D, E\}$ with $A = \hat{0}\hat{t}_2\hat{t}_0\hat{t}_1\hat{t}_3$, $B = \hat{0}\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3$, $C = \hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3$, $D = \hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_3$, $E = \hat{0}\hat{t}_2\hat{t}_0\hat{t}_2\hat{t}_0\hat{t}_0\hat{t}_1\hat{t}_0\hat{t}_1\hat{t}_3$, and $\mathbb{Z}_{-\beta}$ is described by the anti-morphism $\hat{\varphi}_{-\beta} : \hat{R}_\beta^* \rightarrow \hat{R}_\beta^*$ given by

$$\hat{\varphi}_{-\beta} : A \mapsto AB, \quad B \mapsto AC, \quad C \mapsto AD, \quad D \mapsto AED, \quad E \mapsto ABD.$$

The $(-\beta)$ -integers in $[-\beta^3, \beta^4]$ are represented in Figure 4, and we have

$$L(A) = 1, \quad L(B) = \beta - 1, \quad L(C) = \beta^2 - \beta - 1, \quad L(D) = \beta^2 - \beta \approx 2.659, \quad L(E) = \beta.$$

Note that $L(D) > \beta > 2$. Moreover, the cardinality of $\Delta_{-\beta}$ is larger than that of V_β , which in turn is larger than the algebraic degree d of β ($\#\Delta_{-\beta} = 5$, $\#V_\beta = 4$, $d = 3$).

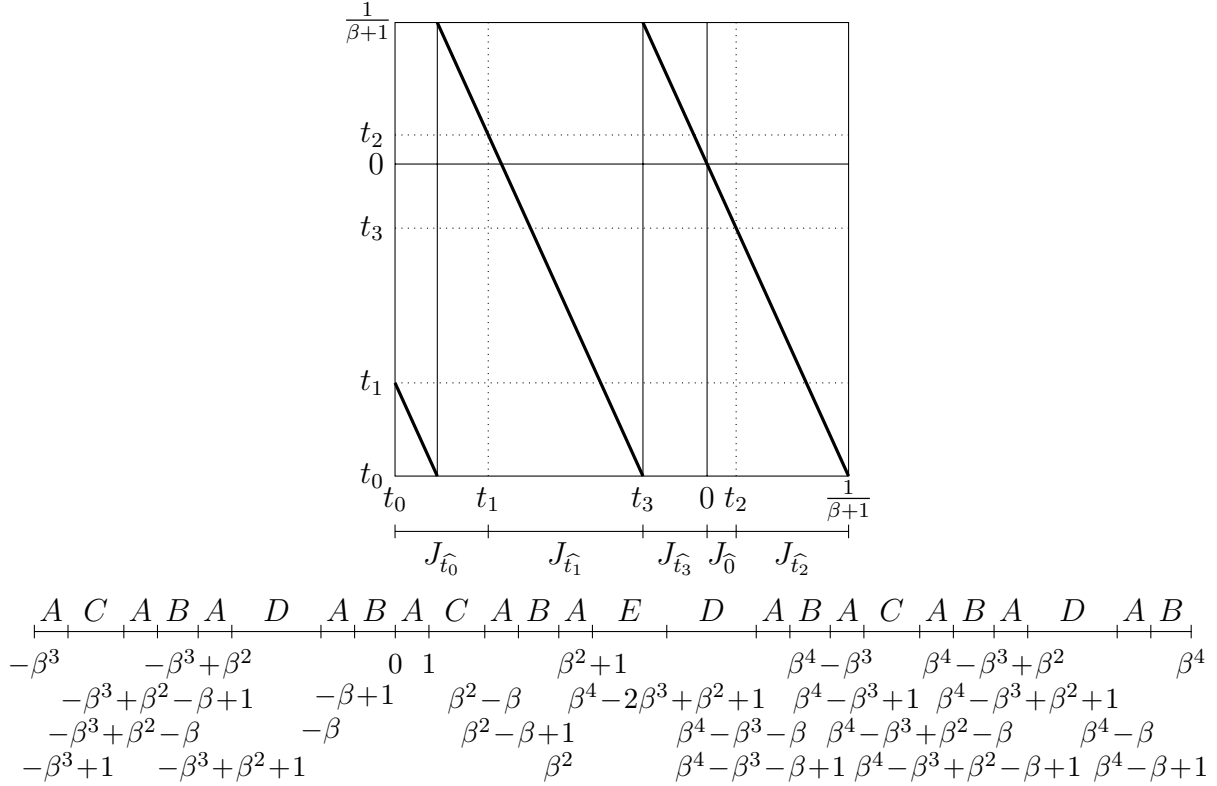


FIGURE 4. The $(-\beta)$ -transformation and $\mathbb{Z}_{-\beta} \cap [-\beta^3, \beta^4]$ from Example 3.

Example 4. Let $\beta > 1$ with $\beta^6 = 3\beta^5 + 2\beta^4 + 2\beta^3 + \beta^2 - 2\beta - 1$, i.e., $\beta \approx 3.695$, then the $(-\beta)$ -transformation is depicted in Figure 5, where $t_n = T_{-\beta}^n(\frac{-\beta}{\beta+1})$. We have $t_5 = \frac{-1}{\beta+1} = t_6$. The anti-morphism $\widehat{\psi}_\beta : \widehat{V}_\beta^* \rightarrow \widehat{V}_\beta^*$ is therefore given by

$$\begin{aligned} \widehat{\psi}_\beta : \quad \widehat{t}_0 &\mapsto \widehat{t}_3 \widehat{t}_5, & \widehat{t}_2 &\mapsto \widehat{t}_4 \widehat{t}_0 \widehat{t}_2, & \widehat{t}_3 &\mapsto \widehat{t}_5 \widehat{t}_1 \widehat{0} \widehat{t}_4 \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1 \widehat{0}, \\ & \widehat{t}_5 &\mapsto \widehat{t}_2 \widehat{t}_3, & \widehat{t}_1 &\mapsto \widehat{0} \widehat{t}_4 \widehat{t}_0, & \widehat{0} &\mapsto \widehat{t}_5 \widehat{t}_1, & \widehat{t}_4 &\mapsto \widehat{t}_0 \widehat{t}_2 \widehat{t}_3. \end{aligned}$$

In order to deal with shorter words, we group the letters forming the words

$$a = \widehat{0} \widehat{t}_4, \quad b = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1, \quad c = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3 \widehat{t}_5, \quad d = \widehat{t}_2 \widehat{t}_3 \widehat{t}_5 \widehat{t}_1, \quad e = \widehat{t}_0 \widehat{t}_2, \quad f = \widehat{t}_4, \quad g = \widehat{t}_0 \widehat{t}_2 \widehat{t}_3, \quad h = \widehat{t}_5 \widehat{t}_1,$$

which correspond to the intervals $J_a = (0, \frac{1}{\beta+1})$, $J_b = (t_0, 0)$, $J_c = (t_0, t_1)$, $J_d = (t_2, 0)$, $J_e = (t_0, t_3)$, $J_f = (t_4, \frac{1}{\beta+1})$, $J_g = (t_0, t_5)$, $J_h = (t_5, 0)$, occurring in iterated images of J_a .

The anti-morphism $\widehat{\psi}_\beta$ acts on these words by

$$\begin{aligned} \widehat{\psi}_\beta : \quad a &\mapsto b, & b &\mapsto ababac, & c &\mapsto dabac, & d &\mapsto ababae, \\ e &\mapsto fc, & f &\mapsto g, & g &\mapsto habac, & h &\mapsto ag. \end{aligned}$$

Since $\widehat{0}$ only occurs at the beginning of a , the return words of $\widehat{0}$ with their $\widehat{\psi}_\beta$ -images are

$$\begin{aligned} ab &\mapsto ab \, ab \, acb, & aed &\mapsto ab \, ab \, aefcb, \\ acb &\mapsto ab \, ab \, acd \, ab \, acb, & aefcb &\mapsto ab \, ab \, acd \, ab \, acgfc b, \\ acd &\mapsto ab \, ab \, aed \, ab \, acb, & acgfc b &\mapsto ab \, ab \, acd \, ab \, \underbrace{acgh}_{=acb} \, ab \, acd \, ab \, acb. \end{aligned}$$

Therefore, $\mathbb{Z}_{-\beta}$ is described by the anti-morphism $\widehat{\varphi}_{-\beta} : \widehat{R}_\beta^* \rightarrow \widehat{R}_\beta^*$ which is defined by

$$\begin{aligned} \widehat{\varphi}_{-\beta} : \quad A &\mapsto AAB, & L(A) &= 1, \\ B &\mapsto AACAB, & L(B) &= \beta - 2 \approx 1.695, \\ C &\mapsto AADAB, & L(C) &= \beta^2 - 3\beta - 1 \approx 1.569, \\ D &\mapsto AAE, & L(D) &= \beta^3 - 3\beta^2 - 2\beta - 1 \approx 1.104, \\ E &\mapsto AACAF, & L(E) &= \beta^4 - 3\beta^3 - 2\beta^2 - \beta - 2 \approx 2.081, \\ F &\mapsto AACABACAB, & L(F) &= \beta^5 - 3\beta^4 - 2\beta^3 - 2\beta^2 + \beta - 2 \approx 3.12. \end{aligned}$$

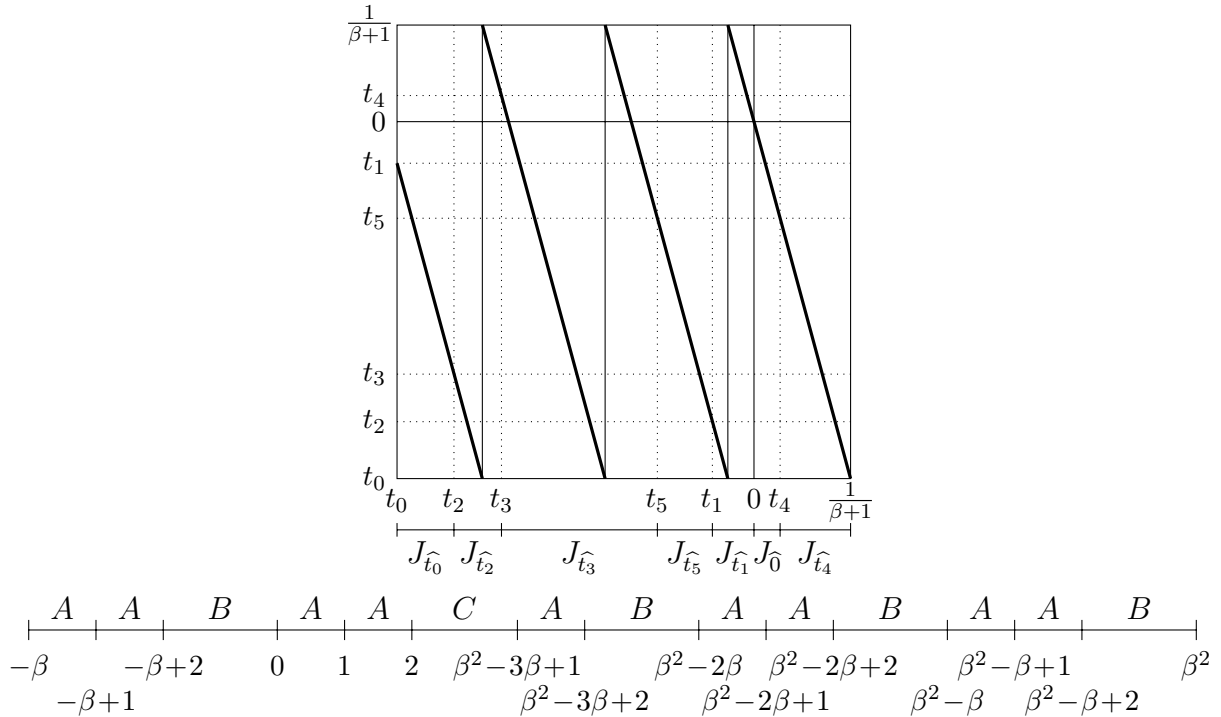


FIGURE 5. The $(-\beta)$ -transformation and $\mathbb{Z}_{-\beta} \cap [-\beta, \beta^2]$ from Example 4.

4. CONCLUSIONS

With every Yrrap number $\beta \geq (1 + \sqrt{5})/2$, we have associated an anti-morphism $\varphi_{-\beta}$ on a finite alphabet. The distances between consecutive $(-\beta)$ -integers are described by a fixed point of $\varphi_{-\beta}$. In [1], the anti-morphism is described explicitly for each $\beta > 1$ such that $T_{-\beta}^n(\frac{-\beta}{\beta+1}) \leq 0$ and $T_{-\beta}^{2n-1}(\frac{-\beta}{\beta+1}) \geq \frac{1-|\beta|}{\beta}$ for all $n \geq 1$. Examples 3 and 4 show that the situation can be quite complicated when this condition is not fulfilled. Although $\varphi_{-\beta}$ can be obtained by a simple algorithm, it seems to be difficult to find a priori bounds for the number of different distances between consecutive $(-\beta)$ -integers or for their maximal value. Only the case of quadratic Pisot numbers β is completely solved; here, we know from [14, 1] that $\#V_\beta = \#\Delta_{-\beta} = 2$.

Recall that the maximal distance between consecutive β -integers is 1, and the number of different distances is equal to the cardinality of the set $\{T_\beta^n(1^-) \mid n \geq 0\}$. Example 3 shows that the $(-\beta)$ -integers do not satisfy similar properties. By generalising Example 4 to $\beta > 1$ with $\beta^6 = (m+1)\beta^5 + m\beta^4 + m\beta^3 + \beta^2 - m\beta - 1$, $m \geq 2$, one sees that the maximal distance can be arbitrarily close to 4 for algebraic integers of degree 6 and $\#V_\beta = 6$.

In a forthcoming paper, we associate anti-morphisms $\varphi_{-\beta}$ on infinite alphabets with non-Yrrap numbers β , by considering the intervals occurring in the iterated $T_{-\beta}$ -images of $(0, \frac{1}{\beta+1})$, cf. Example 4, and we show that the distances between consecutive $(-\beta)$ -integers can be unbounded, e.g. for $\beta > 1$ satisfying $\frac{-\beta}{\beta+1} = \sum_{k=1}^{\infty} a_k(-\beta)^{-k}$ where $a_1a_2 \cdots = 3123212312322 \cdots$ is a fixed point of the morphism $3 \mapsto 31232$, $2 \mapsto 2$, $1 \mapsto 1$. For Yrrap numbers β , this implies that there is no bound for the distance between consecutive $(-\beta)$ -integers which is independent of β . However, large distances occur probably only far away from 0 and when $\#V_\beta$ is large, and it would be interesting to quantify these relations.

Another topic that is worth investigating is the structure of the sets $S_{-\beta}(x)$ for $x \neq 0$, and of the corresponding tilings when β is a Pisot unit. A related question is whether $\mathbb{Z}_{-\beta}$ can be given by a cut and project scheme, cf. [5, 12].

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