

**PRESENTATIONS OF SYMBOLIC DYNAMICAL
SYSTEMS BY DIRECTED LABELLED GRAPHS
(NOTES FOR A "MINI-COURS",
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In symbolic dynamics one studies subshifts (X, S_X) , X a shift-invariant closed subset of the shift space $\Sigma^{\mathbb{Z}}$ and S_X the restriction of the shift on $\Sigma^{\mathbb{Z}}$ to X . An introduction to the theory of subshifts is given in [Ki] and in [LM]. See here also [BP].

The topic of the first talk is a general theory that also includes presentations by compact graphs. The topic of the the second talk are synchronization properties of subshifts that lead to presentations by denumerable graphs.

We fix terminology and notation. Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set

$$x_{[i,k]} = (x_j)_{i \leq j \leq k}, \quad x \in X, i, k \in \mathbb{Z}, i \leq k,$$

and

$$X_{[i,k]} = \{x_{[i,k]} : x \in X\}, i, k \in \mathbb{Z}, i \leq k.$$

We use similar notation also for blocks,

$$b_{[i',k']} = (b_j)_{i' \leq j \leq k'}, \quad b \in X_{[i,k]}, i \leq i' \leq k' \leq k,$$

and also if indices range in semi-infinite intervals. $\mathcal{L}(X)$ is the set of admissible words of X . When convenient we identify blocks with the words they carry. We set

$$\begin{aligned} \Gamma_n^+(a) &= \{b \in X_{(k,k+n]} : (a, b) \in X_{[i,k+n]}\}, & n \in \mathbb{N}, \\ \Gamma_\infty^+(a) &= \{y^+ \in X_{(k,\infty)} : (a, y^+) \in X_{[i,\infty)}\}, \\ \Gamma^+(a) &= \Gamma_\infty^+(a) \cup \bigcup_{n \in \mathbb{N}} \Gamma_n^+(a), & a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k. \end{aligned}$$

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Γ^- has the time symmetric meaning. We set

$$\begin{aligned}\omega_n^+(a) &= \bigcap_{x^- \in \Gamma_\infty^-(a)} \{b \in X_{(k, k+n]} : (x^-, a, b) \in X_{(-\infty, k+n]}\}, \\ \omega_\infty^+(a) &= \bigcap_{x^- \in \Gamma_\infty^-(a)} \{y^+ \in X_{(k, \infty)} : (x^-, a, y^+) \in X\}, \\ \omega^+(a) &= \omega_\infty^+(a) \cup \bigcup_{n \in \mathbb{N}} \omega_n^+(a), \quad a \in X_{[i, k]}, i, k \in \mathbb{Z}, i \leq k.\end{aligned}$$

ω^- has the time symmetric meaning. The unstable set of a point $x \in X$ is denoted by $W_X^-(x)$,

$$W_X^-(x) = \bigcup_{I \in \mathbb{N}} \{y \in X : y_i = x_i, i \leq I\}.$$

The $W_X^-(x), x \in X$, carry the inductive limit topologies of the compact topologies on the sets $\{y \in X : y_i = x_i, i \leq I\}, I \in \mathbb{N}$.

We recall that, given subshifts $X \subset \Sigma^{\mathbb{Z}}, \bar{X} \subset \bar{\Sigma}^{\mathbb{Z}}$, and a topological conjugacy $\varphi : X \rightarrow \bar{X}$, there is for some $L \in \mathbb{Z}_+$ a block mapping

$$\Phi : X_{[-L, L]} \rightarrow \bar{\Sigma}$$

such that

$$\varphi(x) = (\Phi(x_{[i-L, i+L]}))_{i \in \mathbb{Z}}.$$

We say then that φ is given by Φ , and we write

$$\Phi(a) = (\Phi(a_{[j-L, j+L]}))_{i+L \leq j \leq k-L}, \quad a \in X_{[i, k]}, \quad i, k \in \mathbb{Z}, k - i \geq 2L,$$

and use similar notation if indices range in semi-infinite intervals. The interval $[-L, L]$ is called a coding window.

1. PRESENTATIONS OF SUBSHIFTS BY SHANNON GRAPHS

Let Σ be a finite alphabet and consider a directed graph \mathcal{G} with vertex set \mathcal{V} whose directed edges are labeled with symbols in Σ . The graph \mathcal{G} is called Shannon if its labeling is 1-right resolving in the sense that for every vertex $V \in \mathcal{V}$ and for every symbol $\sigma \in \Sigma$ there is at most one edge in \mathcal{G} that leaves V and that carries the label σ . Shannon graphs are also known as deterministic transition systems. Denote the set of vertices $V \in \mathcal{V}$ of a Shannon graph \mathcal{G} that have an outgoing edge that carries the label $\sigma \in \Sigma$ by $\mathcal{V}(\sigma)$, and for $V \in \mathcal{V}(\sigma)$ denote by $\tau_\sigma(V)$ the final vertex of the edge that leaves V and that carries the label σ . We call $(\tau_\sigma)_{\sigma \in \Sigma}$ the transition rule of the Shannon graph. The forward context $\Gamma_\infty^+(V)$ of a vertex V of a Shannon graph \mathcal{G} is defined as the closed subset of $\Sigma^{\mathbb{N}}$ that contains the label sequences of the semi-infinite paths in \mathcal{G} that leave V . A Shannon graph is called forward separated if distinct vertices have distinct forward contexts. Call a Shannon graph compact if its vertex set \mathcal{V} carries a compact topology such that the sets $\mathcal{V}(\sigma)$, are open and such that the mappings $V \rightarrow \tau_\sigma(V), (V \in \mathcal{V}(\sigma)), \sigma \in \Sigma$ are continuous. We say that a Shannon graph \mathcal{G} , in which every vertex has at least one incoming edge and at least one outgoing edge presents a subshift $X \subset \Sigma^{\mathbb{Z}}$ if the

set of admissible words of X coincides with the set of label sequences of the finite paths in \mathcal{G} .

For a finite alphabet Σ denote by $\mathcal{V}(\Sigma)$ the set of closed subsets of $\Sigma^{\mathbb{N}}$ with its compact Hausdorff subset topology. $\mathcal{V}(\Sigma)$ is the vertex set of a compact Shannon graph $\mathcal{G}(\Sigma)$: For $\sigma \in \Sigma$ the set $\mathcal{V}(\Sigma)(\sigma)$ of $\mathcal{G}(\Sigma)$ is equal to the set of $V \in \mathcal{V}$ that contain a sequence that starts with σ , and the transition rule for $\mathcal{G}(\Sigma)$ is

$$(1) \quad \tau_{\sigma}(V) = \{v_{(1,\infty)} : v \in \mathcal{V}, v_1 = \sigma\}, \quad V \in \mathcal{V}(\Sigma)(\sigma), \sigma \in \Sigma.$$

The sets $\mathcal{V}(\Sigma)(\sigma), \sigma \in \Sigma$ are closed and we can associate to the finite alphabet Σ also the topological Markov chain $tM(\Sigma)$ which is the compact dynamical system that is obtained by having the left shift act on the space

$$\{(x_i, V_i)_{i \in \mathbb{Z}} \in (\Sigma \times \mathcal{V}(\Sigma))^{\mathbb{Z}} : V_{i+1} = \tau_{x_{i+1}}(V_i), i \in \mathbb{Z}\}.$$

Denote by $\mathcal{C}(W_{\Sigma^{\mathbb{Z}}}^-(x))$ the set of closed subsets of $W_{\Sigma^{\mathbb{Z}}}^-(x), x \in \Sigma^{\mathbb{Z}}$. For the finite alphabet Σ we introduce further the set \mathcal{C}^{\bullet} of pairs (x, C) , where $x \in \Sigma^{\mathbb{Z}}$ and where $C \in \mathcal{C}(W^-(x))$. Having the left shift act on $\mathcal{C}^{\bullet}(\Sigma)$ one obtains a dynamical system for which one has a shift-commuting bijection ψ_{Σ} onto $tM(\Sigma)$. This bijection ψ_{Σ} assigns to a point $(x, C) \in \mathcal{C}^{\bullet}(\Sigma)$ the point $(x_i, V_i(C))_{i \in \mathbb{Z}} \in tM(\Sigma)$ that is given by

$$V_i(C) = \{y \in C : x_j = y_j, j \leq i\}, \quad i \in \mathbb{Z}.$$

In order to turn $\mathcal{C}^{\bullet}(\Sigma)$ into a compact dynamical system one transports the topology on $tM(\Sigma)$ to $\mathcal{C}^{\bullet}(\Sigma)$ by means of the inverse of the bijection ψ_{Σ} .

We call a subset \mathcal{V} of $\mathcal{V}(\Sigma)$ transition complete if for $V \in \mathcal{V} \cap \mathcal{V}(\Sigma)(\sigma)$ also $\tau_{\sigma}(V) \in \mathcal{V}$. To a transition complete subset \mathcal{V} of $\mathcal{V}(\Sigma)$ we associate the sub-Shannon graph $\mathcal{G}(\mathcal{V})$ of $\mathcal{G}(\Sigma)$ that has as vertex set the set \mathcal{V} and as transition rule the restriction of the rule (1) to \mathcal{V} . The set of forward contexts of the vertices of a forward separated Shannon graph is a transition complete subset of $\mathcal{V}(\Sigma)$, and the mapping that sends every vertex of a forward separated Shannon graph to its forward context is an isomorphism of \mathcal{G} onto the Shannon graph that is associated with the set of forward contexts of \mathcal{G} .

We say that a transition complete subset \mathcal{V} of $\mathcal{V}(\Sigma)$ is retro complete if every $V \in \mathcal{V}$ has a predecessor in \mathcal{V} . For a transition complete compact subset \mathcal{V} of $\mathcal{V}(\Sigma)$ the sets $\mathcal{V}(\sigma), \sigma \in \Sigma$ are also compact, and we set

$$\tau(\mathcal{V}) = \bigcup_{\sigma \in \Sigma} \tau_{\sigma}(\mathcal{V}(\sigma)),$$

and with $\tau^{(0)}(\mathcal{V}) = \tau(\mathcal{V})$, we set inductively

$$\tau^{(n)}(\mathcal{V}) = \tau(\tau^{(n-1)}(\mathcal{V})), \quad n \in \mathbb{N}.$$

Here

$$\tau^{(n)}(\mathcal{V}) \subset \tau^{(n-1)}(\mathcal{V}), \quad n \in \mathbb{N},$$

and the intersection $\bigcap_{n \in \mathbb{N}} \tau^{(n)}(\mathcal{V})$ is the maximal transition and retro complete subset of \mathcal{V} .

The transition and retro complete subsets of $\mathcal{V}(\Sigma)$ are in one-to-one correspondence with the shift invariant subsets of $tM(\Sigma)$: To a transition and retro complete set $\mathcal{V} \subset \mathcal{V}(\Sigma)$ there corresponds the system

$$tM(\mathcal{V}) = \{(x_i, V_i)_{i \in \mathbb{Z}} \in (\Sigma \times \mathcal{V})^{\mathbb{Z}} : V_{i+1} = \tau_{x_{i+1}}(V_i), i \in \mathbb{Z}\}.$$

Also, by assigning to a transition and retro complete subset \mathcal{V} of $\mathcal{V}(\Sigma)$ the dynamical system $C^\bullet(\mathcal{V}) = \psi_\Sigma^{-1}(tM(\mathcal{V}))$ sets up a one-to one correspondence between the transition and retro complete subsets of $\mathcal{V}(\Sigma)$ and the shift invariant subsystems C^\bullet of $C^\bullet(\Sigma)$ with the property that $(x, C) \in C^\bullet$ and $y \in C$ imply that $(y, C) \in C^\bullet$.

Lemma 1. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift. Let $x^- \in X_{(-\infty, 0]}$, $C \in \mathcal{V}(\Sigma)$, and*

$$x^-(k) \in X_{(-\infty, 0]}, \quad C(k) \in \mathcal{V}(\Sigma), C(k) \subset \Gamma_\infty^+(x^-(k)), \quad k \in \mathbb{N},$$

and let

$$x^- = \lim_{k \rightarrow \infty} x^-(k), \quad C = \lim_{k \rightarrow \infty} C(k).$$

Then

$$(2) \quad C \subset \Gamma_\infty^+(x^-).$$

Proof. For $n \in \mathbb{N}$ let $k_n \in \mathbb{N}$ be such that

$$x_{[-n, 0]}^- = x_{[-n, 0]}^-(k), \quad C_{[1, n]} = C_{[1, n]}(k), \quad k \geq k_n.$$

Then for $x^+ \in C$ and $n \in \mathbb{N}$

$$(x_{[-n, 0]}^-, x_{[1, n]}^+) = (x_{[-n, 0]}^-(k), x_{[1, n]}^+(k)) \in X_{[-n, n]}, \quad k \geq k_n,$$

which implies (2) \square

Proposition 1. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift. The set*

$$\mathcal{V}_\circ(X) = \bigcup_{x^- \in X_{(-\infty, 0]}} \{V \in \mathcal{V}(\Sigma) : V \subset \Gamma_\infty^+(x^-)\}$$

is closed.

Proof. Apply the Lemma and the compactness of $X_{(-\infty, 0]}$. \square

For a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set

$$\mathcal{V}_{max}(X) = \bigcap_{n \in \mathbb{N}} \tau^n(\mathcal{V}_\circ(X)),$$

$\mathcal{G}(\mathcal{V}_{max}(X))$ presents X and we refer to $\mathcal{G}(\mathcal{V}_{max}(X))$ as the maximal presenting Shannon graph of X . This terminology is justified by the fact that every transition and retro complete subset V of $\mathcal{V}(\Sigma)$, such that $\mathcal{G}(V)$ presents X , is a subset of $\mathcal{V}_{max}(X)$.

Proposition 2. *Let $X \subset \Sigma^{\mathbb{Z}}, \tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ be subshifts, and let $\varphi : X \rightarrow \tilde{X}$ be a topological conjugacy. Then*

$$(x, C) \rightarrow (\varphi(x), \varphi(C)) \quad ((x, C) \in C^\bullet(\mathcal{V}_{max}(X)))$$

is a topological conjugacy of $C^\bullet(\mathcal{V}_{max}(X))$ onto $C^\bullet(\mathcal{V}_{max}(\tilde{X}))$.

Proof. Let $L \in \mathbb{Z}_+$ be such that $[-L, L]$ is a coding window for φ and φ^{-1} , and let φ be given by the block map $\Phi : X_{[-L, L]} \rightarrow \tilde{\Sigma}$. Let then $x = (x_i)_{i \in \mathbb{Z}} \in X$, and set $\tilde{x} = \varphi(x)$. Also let $(x, C) \in C^\bullet(\mathcal{V}_{max}(X))$, and set $\tilde{C} = \varphi(C)$. Then

$$V_1(\tilde{C}) = \{\Phi(y^+) : y^+ \in V_{-L+1}(C), \Phi(x_{(-3L, -L]}, y_{(-L, L+1]}^+) = \tilde{x}_{(-2L, 0]}\}.$$

This proves the proposition. \square

Given subshifts $X \subset \Sigma^{\mathbb{Z}}, \tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi : X \rightarrow \tilde{X}$, φ will also denote the topological conjugacy that sends the point $(x, C) \in C^\bullet(\mathcal{V}_{max}(X))$ to the point $(\varphi(x), \varphi(C)) \in C^\bullet(\mathcal{V}_{max}(\tilde{X}))$. Given transition and retro complete subsets $\mathcal{V} \subset \mathcal{V}_{max}(X)$ and $\tilde{\mathcal{V}} \subset \mathcal{V}_{max}(\tilde{X})$ where $\mathcal{G}(\mathcal{V})$ presents X and $\mathcal{G}(\tilde{\mathcal{V}})$ presents \tilde{X} , if $\varphi(C^\bullet(\mathcal{V})) = C^\bullet(\tilde{\mathcal{V}})$, then we write also $\varphi(\mathcal{V}) = \tilde{\mathcal{V}}$ and $\varphi(tM(\mathcal{V})) = tM(\tilde{\mathcal{V}})$.

Given a construction for a subshift $X \subset \Sigma^{\mathbb{Z}}$ of a transition and retro complete subset $\mathcal{V}_X \subset \mathcal{V}_{max}(X)$ such that $\mathcal{G}(\mathcal{V}_X)$ presents X , we say that \mathcal{V}_X is canonical, if for subshifts $X \subset \Sigma^{\mathbb{Z}}, \tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi : X \rightarrow \tilde{X}$, $\varphi(\mathcal{V}_X) = \mathcal{V}_{\tilde{X}}$. \mathcal{V}_{max} itself is canonical. The standard example of the canonical situation is here

$$\mathcal{V}_{standard}(X) = \{\Gamma_\infty^+(x^-) : x^- \in X_{(-\infty, 0]}\}.$$

It is

$$tM(\mathcal{V}_{standard}(X)) = \{(x_i, \Gamma_\infty^+((x_j)_{j < i}))_{i \in \mathbb{Z}} : (x_i)_{i \in \mathbb{Z}} \in X\},$$

and

$$C^\bullet(\mathcal{V}_{standard}(X)) = \{(x, W^-(X)) : x \in X\},$$

and from this it is seen that $\mathcal{V}_{standard}$ is canonical. The finiteness of $\mathcal{V}_{standard}$ characterizes the sofic case [W]. That $\mathcal{V}_{standard}$ is canonical was first noted in the sofic case in [Kr1, Kr2]. For a subshift $X \subset \Sigma^{\mathbb{Z}}$ the closure of $\mathcal{V}_{standard}(X)$ is a compact Shannon graph that presents X . This presentation appears in Matsumoto's theory of λ -graph systems (see [M, KM]). Also in non-sofic cases $\mathcal{V}_{standard}(X)$ itself can be compact. For instance, the coded system (see [BH]) with alphabet $\Sigma = \{\gamma, 0, 1\}$ and code $\{\gamma 0^l : l \in \mathbb{N}\}$ has a compact $\mathcal{V}_{standard}$, while for the coded system with alphabet $\Sigma = \{\gamma, 0, 1\}$ and code $\{\gamma 0^{2l} : l \in \mathbb{N}\}$ $\mathcal{V}_{standard}$ is not compact. The content of the following proposition is that the compactness of $\mathcal{V}_{standard}$ is an invariant of topological conjugacy.

Proposition 3. *Let $X \subset \Sigma^{\mathbb{Z}}$ and $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ be topologically conjugate subshifts, and let $\mathcal{V}_{standard}(X)$ be compact. Then $\mathcal{V}_{standard}(\tilde{X})$ is also compact.*

Proof. This follows from Proposition 2. \square

Another example is the word Shannon graph of a subshift $X \subset \Sigma^{\mathbb{Z}}$,

$$\mathcal{V}_{word}(X) = \{\{x^+\} : x^+ \in X_{[1, \infty)}\}.$$

It is

$$tM(\mathcal{V}_{word}(X)) = \{(x_i, \{(x_j)_{j \geq i}\})_{i \in \mathbb{Z}} : (x_i)_{i \in \mathbb{Z}} \in X\},$$

and

$$C^\bullet(\mathcal{V}_{word}(X)) = \{(x, \{x\}) : x \in X\}.$$

\mathcal{V}_{word} is compact.

2. NOTIONS OF SYNCHRONIZATION

We describe synchronizing shifts and their synchronizing Shannon graphs and then consider the more general notions of s-synchronization and a-synchronization.

2.1 SYNCHRONIZATION

A word $b \in \mathcal{L}(X)$ is called a synchronizing word of the topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ if for $c \in \Gamma^-(b), d \in \Gamma^+(b)$ one has that $cbd \in \mathcal{L}(X)$. Equivalently, a synchronizing word of X can be defined as a word $b \in \mathcal{L}(X)$ such that $\Gamma^-(b) = \omega^-(b)$, or, such that $\Gamma^+(b) = \omega^+(b)$. We denote the set of synchronizing words of a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ by $\mathcal{L}_{synchron}(X)$.

Lemma 2. *Let $X \subset \Sigma^{\mathbb{Z}}, \tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ be topologically transitive subshifts and let $L \in \mathbb{Z}_+$ be such that there is a topological conjugacy of X onto \tilde{X} that has together with its inverse coding window $[-L, L]$ with its inverse is given by a block map*

$$\tilde{\Phi} : \tilde{X}_{[-L, L]}.$$

Let b be a synchronizing word for X and let $\tilde{b} \in \mathcal{L}(\tilde{X})$ be a word such that $B = \Phi(\tilde{b})$. Then (\tilde{b}) is a synchronizing word for \tilde{X} .

Proof. Let $\tilde{c} \in \Gamma^-(\tilde{b}), \tilde{d} \in \Gamma^+(\tilde{b})$ and choose words $\tilde{c}' \in \Gamma^-(\tilde{c}\tilde{b})$ and $\tilde{d}' \in \Gamma^+(\tilde{b}\tilde{d})$ of length $2L$. With words $c \in \Gamma^-(b), d \in \Gamma^+(b)$, that are given by

$$\tilde{\Phi}(\tilde{c}'\tilde{c}\tilde{b}) = cb, \quad \tilde{\Phi}(\tilde{b}\tilde{d}\tilde{d}') = bd$$

one has

$$cbd \in \mathcal{L}(X),$$

and

$$\tilde{c}\tilde{b}\tilde{d} = \Phi(cbd). \quad \square$$

A topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ with a synchronizing word is called synchronizing (see [BH]). For $a \in \mathcal{L}_{synchron}(X)$ and $b \in \Gamma^+(a)$ also $ab \in \mathcal{L}_{synchron}(X)$. It follows for a synchronizing subshift X that the set

$$\mathcal{V}_{synchron}(X) = \{\Gamma_\infty^+(a) : a \in \mathcal{L}_{synchron}(X)\}$$

is the vertex set of an irreducible Shannon graph that presents X . $\mathcal{V}_{synchron}(X)$ was first constructed in the sofic case in [F], where the term "Shannon graph" was introduced. It follows from Lemma 2 that synchronization is an invariant of topological conjugacy. The presentation of a synchronizing subshift by its synchronizing Shannon graph is canonical. This is the content of the following proposition.

Theorem 1. *Let $X \subset \Sigma^{\mathbb{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ be synchronizing subshifts, and let $\varphi : X \rightarrow \tilde{X}$ be a topological conjugacy. Then*

$$\varphi(\mathcal{V}_{\text{synchronro}}(X)) = \mathcal{V}_{\text{synchronro}}(\tilde{X}).$$

Proof. Let $L \in \mathbb{Z}_+$ such that φ has the coding window $[-L, L]$. Let there be given a point

$$(C(i), x_i)_{i \in \mathbb{Z}} \in C^\bullet(\mathcal{G}_{\text{synchronro}}(X)).$$

Let b be a synchronizing word of X such that

$$(1) \quad C(-2L) = \Gamma_\infty^+(b),$$

and let $d \in \Gamma^+(b)$ be such that $bd \in \Gamma^-(b)$. Let $x' \in X$ be such that $X_{(-\infty, -2L)}$ carries the infinite concatenation of db . and such that

$$(2) \quad x'_{[-2L, \infty)} = x_{[-2L, \infty)}.$$

Setting

$$\tilde{x} = \varphi(x), \quad \tilde{x}' = \varphi(x'),$$

one has from (2) that

$$(3) \quad x'_{[-L, \infty)} = x_{[-L, \infty)},$$

and setting

$$(\tilde{C}(i), \tilde{x}_i)_{i \in \mathbb{Z}} = \varphi((C(i), x_i)_{i \in \mathbb{Z}}),$$

and

$$C'(i) = \Gamma_\infty(x'_{(-\infty, i)}), \quad i \in \mathbb{Z}.$$

$$(\tilde{C}'(i), \tilde{x}'_i)_{i \in \mathbb{Z}} = \varphi((C'(i), x'_i)_{i \in \mathbb{Z}}),$$

one has from (1) and (3) that

$$\tilde{C}'(1) = \tilde{C}(1).$$

It follows from the Lemma that

$$\tilde{C}'(1) \in \mathcal{V}_{\text{synchronro}}(\tilde{X}),$$

and it is shown that

$$\tilde{C}'(i) \in \mathcal{V}_{\text{synchronro}}(\tilde{X}), \quad i \in \mathbb{Z}. \quad \square$$

2.2 S-SYNCHRONIZATION

A word b that is admissible for a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ is called an s-synchronizing word of X , if for all $c \in \mathcal{L}(X)$ there exists a $d \in \Gamma^+(c)$ such that $cd \in \omega^-(b)$. A synchronizing word is s-synchronizing. We denote the set of s-synchronizing words of a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ by $\mathcal{L}_{s\text{-synchronro}}(X)$. We note that for an s-synchronizing word b there exists in particular a word $d \in \Gamma^+(b)$ such that $bd \in \omega^-(b)$ and this implies that $\omega_\infty^-(b) \neq \emptyset$.

Lemma 3. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a topologically transitive subshift, and let $b \in \mathcal{L}(X)$ be an s-synchronizing word for X . Let $d \in \mathcal{L}(X)$ be such that $bd \in \omega^-(b)$, set $a = bd$, and let l denote the length of a . Let $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ be a subshift that is topologically conjugate to X , and let $L \in \mathbb{N}$ be such that $[-lL, lL]$ is a coding window of a topological conjugacy of X onto \tilde{X} , that is given by the block map*

$$\Phi : X_{[-lL, lL]} \rightarrow \tilde{\Sigma}.$$

Then $\Phi(a^{3L}b)$ is an s-synchronizing word for \tilde{X} .

Proof. Let $\tilde{c} \in \mathcal{L}(\tilde{X})$ and let $c \in \mathcal{L}(X)$ be such that $\tilde{c} = \Phi(c)$. Let then $\tilde{d} \in \Gamma^+(\tilde{c})$ be given by

$$\tilde{c}\tilde{d} = \Phi(cda^{3L}b).$$

Then

$$\tilde{c}\tilde{d} \in \omega^-(\tilde{b}). \quad \square$$

It follows from Lemma 3 that s-synchronization is an invariant of topological conjugacy. The presentation of an s-synchronizing shift by its s-synchronizing Shannon graph is invariantly associated to the subshift. This is shown in the same way as for the presentation of a synchronizing shift by its synchronizing Shannon graph, Lemma 3 taking the place of Lemma 2.

2.3 A-SYNCHRONIZATION

A word b that is admissible for a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ is called an a-synchronizing word for X if B satisfies the following two conditions a-s(1) and a-s(2):

Condition a-s(1). *For all $c \in \mathcal{L}(X)$ there exists a $d \in \Gamma^-(c)$ such that $dc \in \omega^+(b)$.*

Condition a-s(2). *For all $c \in \omega_{\circ}^+(b)$ there exists a $d \in \Gamma^+(c) \cap \Gamma^-(b)$ such that $cdb \in \omega^+(b)$ and $\omega_{\infty}^+(bcd) = \omega_{\infty}^+(b)$.*

Condition a-s(1) is equivalent to requiring that the word b is an s-synchronizing word for the inverse of the subshift $X \subset \Sigma^{\mathbb{Z}}$.

Lemma 4. *Let b be an a-synchronizing word of the topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$. Then there exists a word $d \in \Gamma^-(b)$ such that $db \in \omega^+(b)$ and $\omega_{\infty}^+(bdb) = \omega_{\infty}^+(b)$.*

Proof. First choose by a-s(1) a word $d' \in \Gamma^-(b)$ such that $d'b \in \omega^+(b)$. Then choose by a-s(2) a word $d'' \in \Gamma^+(d'b) \cap \Gamma^-(b)$ such that $d'bd''b \in \omega^+(b)$ and such that $\omega_{\infty}^+(bd'bd''b) = \omega_{\infty}^+(b)$. Set $d = d'bd''$. \square

From the preceding lemma it is seen that the formulation of the following lemma is meaningful.

Lemma 5. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a topologically transitive subshift, and let $b \in \mathcal{L}(X)$ be an a-synchronizing word for X . Let $d \in \mathcal{L}(X)$ be such that $db \in \omega^+$ and $\omega_{\infty}^+(bdb) = \omega_{\infty}^+(b)$. Set $a = db$, and let l denote the length of a . Let $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ be a subshift that is topologically conjugate to X , and let $L \in \mathbb{N}$ be such that $[-lL, lL]$ is a coding window of both, a topological conjugacy and of its inverse that the topological*

conjugacy given by a block map $\Phi : X_{[-lL, lL]} \rightarrow \tilde{\Sigma}$, with the inverse given by a block map $\tilde{\Phi} : \tilde{X}_{[-lL, lL]} \rightarrow \Sigma$. Then $\tilde{b} = \Phi(ba^{3L+1})$ is an a -synchronizing word for \tilde{X} .

Proof. That \tilde{b} satisfies a-s(1) is Lemma 4. For the proof that \tilde{b} also satisfies a-s(2), let $\tilde{x}^+ \in \omega_\infty^+(\tilde{b})$ and define an $x^+ \in \omega_\infty^+(b)$ by

$$(b, x) = \tilde{\Phi}(\tilde{b}, \tilde{x}^+).$$

Let then $n \in \mathbb{N}$, denote the word that is carried by the block $_{[1, n]}$ by \tilde{c} , and denote the word that is carried by the block $x_{[1, n+(L+1)l]}$ by c . By hypothesis the word b satisfies a-s(2) and therefore one can find a word $d \in \mathcal{L}(X)$ such that

$$(1) \quad cdb \in \omega^+(b),$$

and

$$(2) \quad \omega_\infty^+(bcd) = \omega_\infty^+(b).$$

Define a word $\tilde{d} \in \mathcal{L}(\tilde{X})$ by

$$\Phi(ba^{3L+1}cdba^{3L+1}) = \tilde{b}\tilde{c}\tilde{d}.$$

(1) implies that

$$\tilde{c}\tilde{d}\tilde{b} \in \omega^+(\tilde{b}),$$

and (2) implies that

$$\tilde{b}\tilde{c}\tilde{d}\tilde{b} \in \omega_\infty^+(b). \quad \square$$

For an a -synchronizing word b of the topologically transitive subshift $X \subset \Sigma^\mathbb{Z}$ and for a word $a \in \Gamma^+(b)$ the word ba is again a -synchronizing for X . It follows that the set

$$\mathcal{V}_{a\text{-synchro}}(X) = \{\Gamma_\infty^+(b) : b \in \mathcal{L}(X)\}$$

is the vertex set of a Shannon graph that we denote by $\mathcal{G}_{a\text{-synchro}}(X)$. $\mathcal{G}_{a\text{-synchro}}(X)$ is the union of its irreducible components all of which present X . One can see from condition (b) of Proposition that the number of irreducible component of $\mathcal{G}_{a\text{-synchro}}(X)$ is an invariant of topological conjugacy. In an attempt to maintain an analogy with synchronization and s -synchronization we say that a topologically transitive subshift $X \subset \Sigma^\mathbb{Z}$ is a -synchronizing, if X has a a -synchronizing word and if $\mathcal{G}_{a\text{-synchro}}(X)$ is irreducible. The presentation of a synchronizing subshift by its synchronizing Shannon graph is canonical. This is the content of the following proposition.

Theorem 2. *Let $X \subset \Sigma^\mathbb{Z}$, $\tilde{X} \subset \tilde{\Sigma}^\mathbb{Z}$ be a -synchronizing subshifts, and let $\varphi : X \rightarrow \tilde{X}$ be a topological conjugacy. Then*

$$\varphi(\mathcal{V}_{a\text{-synchro}}(X)) = \mathcal{V}_{a\text{-synchro}}(\tilde{X}).$$

Proof. Let $L \in \mathbb{Z}_+$ be such that the topological conjugacy φ has the coding window $[-L, L]$. Let there be given a point

$$(C(i), x_i)_{i \in \mathbb{Z}} \in \mathcal{C}^\bullet(\mathcal{G}_{a\text{-synchro}}(X)).$$

Let b be an a -synchronizing word of X such that

$$(1) \quad C(-2L) = \omega_{\infty}^{+}(b).$$

By Lemma 4 one can choose a $d \in \Gamma^{-}(b)$ such that

$$db \in \omega^{+}(b),$$

and such that

$$\omega_{\infty}^{+}(bdb) = \omega_{\infty}^{+}(b).$$

Let $x' \in X$ be such that $x_{(-\infty, -2L)}$ carries the infinite concatenation of db , and such that

$$(2) \quad x'_{[-2L, \infty)} = x_{[-2L, \infty)}.$$

Setting

$$\tilde{x} = \varphi(x), \tilde{x}' = \varphi(x'),$$

one has from (2) that

$$x'_{[-L, \infty)} = x_{[-L, \infty)},$$

and setting

$$(\tilde{C}(i), \tilde{x}_i)_{i \in \mathbb{Z}} = \varphi((C(i), x_i)_{i \in \mathbb{Z}}),$$

and

$$C'(i) = \bigcup_{n \in \mathbb{N}} \omega^{+}(x'_{[i-n, i]}), \quad i \in \mathbb{Z},$$

$$(\tilde{C}'(i), \tilde{x}'_i)_{i \in \mathbb{Z}} = \varphi((C'(i), x'_i)_{i \in \mathbb{Z}}),$$

one has from (1) and (3) that

$$\tilde{C}'(1) = \tilde{C}(1).$$

It follows from Lemma 5 that

$$\tilde{C}'(1) \in \mathcal{V}_{a\text{-synchronro}}(\tilde{X}),$$

and it is shown that

$$\tilde{C}'(i) \in \mathcal{V}_{a\text{-synchronro}}(\tilde{X}), \quad i \in \mathbb{Z}. \quad \square$$

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