

Dynamical properties of the $(-\beta)$ -transformation

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Aussois, 06/04/2010

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Introduction

I. The $(-\beta)$ -transformation

Define $T_{-\beta} : (0, 1] \rightarrow (0, 1]$ by

$$T_{-\beta}(x) := -\beta x + \lfloor \beta x \rfloor + 1.$$

Let

$$d_{-\beta,1}(x) = \lfloor \beta x \rfloor + 1, \quad d_{-\beta,n}(x) = d_{-\beta,1}(T_{-\beta}^{n-1}(x)) \quad \text{for } n \geq 1.$$

Then

$$x = \frac{-d_{-\beta,1}}{-\beta} + \frac{-d_{-\beta,2}}{(-\beta)^2} + \frac{-d_{-\beta,3}}{(-\beta)^3} + \frac{-d_{-\beta,4}}{(-\beta)^4} + \dots.$$

The sequence $d_{-\beta}(x) = d_{-\beta,1}(x)d_{-\beta,2}(x)\dots$ is called the $(-\beta)$ -expansion of x .

II. Remarks about the definition

- **Ito and Sadahiro 2009** : On the interval $[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$:

$$x \mapsto -\beta x - \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor.$$

→ conjugate to our $T_{-\beta}$ through the conjugacy function $\phi(x) = \frac{1}{\beta+1} - x$. So all results can be translated to our case.

- Our definition is one case of generalized β -transformation studied by **Góra 2007** and **Faller 2008 (Ph.D Thesis)**.

III. Admissible sequence and $(-\beta)$ -shift

A sequence $a_1 a_2 \dots$ is said **admissible** if $\exists x \in (0, 1]$, $d_{-\beta}(x) = a_1 a_2 \dots$.

Alternate order : $a_1 a_2 \dots \prec b_1 b_2 \dots$ if and only if

$$\exists k \geq 1, \quad a_i = b_i \text{ for } i < k \quad \text{and} \quad (-1)^k (b_k - a_k) < 0.$$

Denote $a_1 a_2 \dots \preceq b_1 b_2 \dots$ if $a_1 a_2 \dots \preceq b_1 b_2 \dots$ or $a_1 a_2 \dots = b_1 b_2 \dots$.

The $(-\beta)$ -shift $S_{-\beta}$ on the alphabet $\{1, \dots, \lfloor \beta \rfloor + 1\}$ is the closure of the set of admissible sequences.

Define

$$\begin{aligned} d_{-\beta}^*(0) &:= \lim_{x \rightarrow 0^+} d_{-\beta}(x) \\ &= \begin{cases} \overline{1b_1 b_2 \dots b_{q-1} (b_q - 1)}, & \text{if } d_{-\beta}(1) = \overline{b_1 \dots b_{q-1} b_q} \text{ for some odd } q \\ 1d_{-\beta}(1) & \text{otherwise.} \end{cases} \end{aligned}$$

IV. Admissible sequence and $(-\beta)$ -shift (continue)

Theorem (Ito-Sadahiro 2009)

A sequence $a_1 a_2 \cdots$ is admissible if and only if for each $n \geq 1$

$$d_{-\beta}^*(0) \prec a_n a_{n+1} \cdots \preceq d_{-\beta}(1).$$

A sequence $a_1 a_2 \cdots$ is in $S_{-\beta}$ if and only if for each $n \geq 1$

$$d_{-\beta}^*(0) \preceq x_n x_{n+1} \cdots \preceq d_{-\beta}(1).$$

Theorem (Ito-Sadahiro 2009)

The $(-\beta)$ -shift is **sofic** if and only if $d_{-\beta}(1)$ is **eventually periodic**.

Theorem (Frougny-Lai 2010)

The $(-\beta)$ -shift is of **finite type** if and only if $d_{-\beta}(1)$ is **purely periodic**.

Theorem (Frougny-Lai 2010)

If β is a **Pisot number**, then the $(-\beta)$ -shift is **sofic**.

Dynamical properties of $(-\beta)$ -transformation

I. Some notions of dynamical systems

Suppose $T : X \rightarrow X$ be a dynamical system.

- **exactness** : T acting on (X, \mathcal{B}, μ) is called exact if

$$\bigcap_{n=0}^{\infty} T^{-n} \mathcal{B} = \{X, \emptyset\}$$

or equivalently, for any positive measure set A with $T^n(A) \in \mathcal{B}$ ($n \geq 0$),

$$\mu(T^n(A)) \rightarrow 1 \quad (n \rightarrow \infty.)$$

- **maximal entropy measure** : the measure attaining the maximum of

$$\sup\{h_\mu : \mu \text{ invariant}\}.$$

- **intrinsic ergodicity** : the maximal entropy measure is unique.

II. General piecewise monotone transformation

$T : [0, 1] \rightarrow [0, 1]$.

- finite partition of $[0, 1] : P = \{P_1, \dots, P_N\}$.
- on each P_i , T is monotonic, Lipschitz continuous and $|T'| \geq \rho > 1$.

Lasota-Yorke 1974 : There is an invariant measure $d\mu = h d\lambda$, where $d\lambda$ is the Lebesgue measure and h is a density of bounded variation.

Keller 1978 : The set $\{h \neq 0\}$ is a **finite** union of intervals.

Wagner 1979 : We can decompose $[0, 1] = \cup_{i=1}^s A_i \cup B$, such that

- on each A_i there is an invariant measure which is equivalent to the Lebesgue measure restricted to A_i
- each A_i can be decomposed as $A_i = \cup_{j=1}^{m_i} A_{ij}$ and T^{m_i} is **exact** on each A_{ij} .
- the set B satisfies $T^{-1}B \subset B$ and $\lim_{n \rightarrow \infty} \lambda(T^{-n}B) = 0$.

Hofbauer 1981 : The number of maximal entropy measures is **finite**. If T is topological transitive, then it is **intrinsic ergodic**.

III. Dynamical properties - the $(-\beta)$ case

Ito-Sadahiro 2009 : Let $h_{-\beta}$ be a real-valued function defined on $(0, 1]$ by

$$h_{-\beta}(x) = \sum_{n \geq 1, T_{-\beta}^n(1) \geq x} \frac{1}{(-\beta)^n}.$$

Then the measure $h_{-\beta}(x)d\lambda$ is an invariant measure of $T_{-\beta}$.

Remark : The density may be zero on some intervals. So the invariant measure is not equivalent to the Lebesgue measure. (**Different to the β case**).

Góra 2007 : for $\beta > \gamma_1$ (the smallest Pisot number), the transformation $T_{-\beta}$ is exact and he conjectured that this would hold for all $\beta > 1$.

Faller 2008 : $\beta > \sqrt[3]{2}$, $T_{-\beta}$ admits a unique maximal measure.

IV. Our results-Notations

For each $n \geq 0$, let γ_n be the positive real number defined by

$$\gamma_n^{g_n+1} = \gamma_n + 1, \quad \text{with} \quad g_n = \lfloor 2^{n+2}/3 \rfloor.$$

Then

$$2 > \gamma_0 > \gamma_1 > \gamma_2 > \cdots > 1.$$

Note that γ_0 is the **golden ratio** and that γ_1 is the **smallest Pisot number**.

For each $n \geq 0$ and $1 < \beta < \gamma_n$, set

$$\mathcal{G}_n(\beta) = \{G_{m,k}(\beta) \mid 0 \leq m \leq n, 0 \leq k < g'_{m-1}\}, \quad \text{with} \quad g'_m = \lfloor \frac{2^{m+2} + 1}{3} \rfloor.$$

with open intervals

$$G_{m,k}(\beta) = \begin{cases} (T_{-\beta}^{2^{m+1}+k}(1), T_{-\beta}^{g'_m+k}(1)) & \text{if } k \text{ is even,} \\ (T_{-\beta}^{g'_m+k}(1), T_{-\beta}^{2^{m+1}+k}(1)) & \text{if } k \text{ is odd.} \end{cases}$$

IV. Our results-Theorems

We call an interval a gap if the density of the invariant measure is zero on it.

Theorem (Liao-Steiner 2010)

If $\beta \geq \gamma_0$, then there is no gap. If $\gamma_{n+1} \leq \beta < \gamma_n$, $n \geq 0$, then the set of gaps is $\mathcal{G}_n(\beta)$ which consists of $g_n = \lfloor 2^{n+2}/3 \rfloor$ disjoint non-empty intervals.

Define

$$G(\beta) = \begin{cases} \emptyset & \text{if } \beta \geq \gamma_0, \\ \bigcup_{I \in \mathcal{G}_n(\beta)} I & \text{if } \gamma_{n+1} \leq \beta < \gamma_n, n \geq 0. \end{cases}$$

Theorem (Liao-Steiner 2010)

The transformation $T_{-\beta}$ is topological mixing on $(0, 1] \setminus G(\beta)$,

$$T_{-\beta}^{-1}(G(\beta)) \subset G(\beta) \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda(T_{-\beta}^{-n}(G(\beta))) = 0.$$

V. Our results-Theorems (continue)

Define a morphism on the symbolic space $\{1, 2\}^{\mathbb{N}}$ by

$$\varphi : 1 \mapsto 2, \quad 2 \mapsto 211.$$

Theorem (Liao-Steiner 2010)

For every $n \geq 0$, we have

$$d_{-\gamma_n}(1) = \varphi^n(21^\omega).$$

Hence

$$\lim_{\beta \rightarrow 1} d_{-\beta}(1) = \lim_{n \rightarrow \infty} \varphi^n(2) = 2112221121121122 \dots$$

Remark : $g'_m = |\varphi^m(2)| = |\varphi^{m+1}(1)|$ and $2^{m+1} = |\varphi^m(21)|$.

VI. Our results-Corollaries

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ is exact.

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ has a unique maximal measure, and hence is intrinsic ergodic.

Proofs : From the topological mixing : for each interval $J \subset (0, 1] \setminus G(\beta)$, there is $n \geq 1$ such that $f^n(J) = (0, 1] \setminus G(\beta)$.

VII. Our results-Proofs

For every word $a_1 \cdots a_n \in \{1, 2\}^n$, $n \geq 0$, define the polynomial

$$P_{a_1 \cdots a_n} = (-X)^n + \sum_{k=1}^n a_k (-X)^{n-k} \in \mathbb{Z}[X]$$

Lemma

For $1 \leq m < n$, we have

$$P_{a_1 \cdots a_n} = (-X)^{n-m} (P_{a_1 \cdots a_m} - 1) + P_{a_{m+1} \cdots a_n}.$$

For $n \geq 0$ we have the identities :

- $X^{\frac{1+(-1)^n}{2}} P_{\varphi^n(2)} + X^{\frac{1-(-1)^n}{2}} P_{\varphi^n(11)} = X + 1 = X^{\frac{1+(-1)^n}{2}} + X^{\frac{1-(-1)^n}{2}}$
- $1 - P_{\varphi^n(1)} = X^{\frac{1+(-1)^n}{2}} \prod_{k=0}^{n-1} (X^{|\varphi^k(1)|} - 1)$
- $P_{\varphi^n(21)} - P_{\varphi^n(2)} = (X^{g_n+1} - X - 1) \prod_{k=0}^{n-1} (X^{|\varphi^k(1)|} - 1)$

Thank you for your attention !