Dynamical properties of the \((-\beta)\)-transformation

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Introduction

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I. The \((-\beta)\)-transformation

Define \(T_{-\beta} : (0, 1] \rightarrow (0, 1] \) by

\[ T_{-\beta}(x) := -\beta x + \lfloor \beta x \rfloor + 1. \]

Let

\[ d_{-\beta,1}(x) = \lfloor \beta x \rfloor + 1, \quad d_{-\beta,n}(x) = d_{-\beta,1}(T_{-\beta}^{n-1}(x)) \quad \text{for } n \geq 1. \]

Then

\[ x = \frac{-d_{-\beta,1}}{-\beta} + \frac{-d_{-\beta,2}}{(-\beta)^2} + \frac{-d_{-\beta,3}}{(-\beta)^3} + \frac{-d_{-\beta,4}}{(-\beta)^4} + \cdots. \]

The sequence \(d_{-\beta}(x) = d_{-\beta,1}(x)d_{-\beta,2}(x) \cdots\) is called the \((-\beta)\)-expansion of \(x\).
II. Remarks about the definition

- **Ito and Sadahiro 2009**: On the interval $\left[ -\frac{\beta}{\beta+1}, \frac{1}{\beta+1} \right)$:

  $$x \mapsto -\beta x - \left\lfloor -\beta x + \frac{\beta}{\beta+1} \right\rfloor.$$

  $\rightarrow$ conjugate to our $T_{-\beta}$ through the conjugacy function $\phi(x) = \frac{1}{\beta+1} - x$. So all results can be translated to our case.

- Our definition is one case of generalized $\beta$-transformation studied by **Góra 2007** and **Faller 2008 (Ph.D Thesis)**.
III. Admissible sequence and \((-\beta)\)-shift

A sequence \(a_1a_2\cdots\) is said **admissible** if \(\exists x \in (0, 1], d_{-\beta}(x) = a_1a_2\cdots\).

**Alternate order**: \(a_1a_2\cdots \prec b_1b_2\cdots\) if and only if

\[
\exists k \geq 1, \quad a_i = b_i \text{ for } i < k \quad \text{and} \quad (-1)^k(b_k - a_k) < 0.
\]

Denote \(a_1a_2\cdots \preceq b_1b_2\cdots\) if \(a_1a_2\cdots \preceq b_1b_2\cdots\) or \(a_1a_2\cdots = b_1b_2\cdots\).

The **\((-\beta)\)-shift** \(S_{-\beta}\) on the alphabet \(\{1, \ldots, \lfloor\beta\rfloor + 1\}\) is the closure of the set of admissible sequences.

Define

\[
d^*_{-\beta}(0) := \lim_{x \to 0^+} d_{-\beta}(x)
\]

\[
= \begin{cases} 
1b_1b_2\cdots b_{q-1}(b_q - 1), & \text{if } d_{-\beta}(1) = b_1\cdots b_{q-1}b_q \text{ for some odd } q \\
1d_{-\beta}(1) & \text{otherwise.}
\end{cases}
\]
### IV. Admissible sequence and \((-\beta)\)-shift (continue)

**Theorem (Ito-Sadahiro 2009)**

A sequence \(a_1 a_2 \cdots\) is admissible if and only if for each \(n \geq 1\)

\[
d^*_\beta(0) < a_n a_{n+1} \cdots \leq d_{-\beta}(1).
\]

A sequence \(a_1 a_2 \cdots\) is in \(S_{-\beta}\) if and only if for each \(n \geq 1\)

\[
d^*_\beta(0) \leq x_n x_{n+1} \cdots \leq d_{-\beta}(1).
\]

**Theorem (Ito-Sadahiro 2009)**

The \((-\beta)\)-shift is **sofic** if and only if \(d_{-\beta}(1)\) is eventually periodic.

**Theorem (Frougny-Lai 2010)**

The \((-\beta)\)-shift is of **finite type** if and only if \(d_{-\beta}(1)\) is purely periodic.

**Theorem (Frougny-Lai 2010)**

If \(\beta\) is a **Pisot number**, then the \((-\beta)\)-shift is **sofic**.
Dynamical properties of \((-\beta)\)-transformation
I. Some notions of dynamical systems

Suppose $T : X \to X$ be a dynamical system.

- **exactness**: $T$ acting on $(X, \mathcal{B}, \mu)$ is called exact if
  \[
  \bigcap_{n=0}^{\infty} T^{-n} \mathcal{B} = \{X, \emptyset\}
  \]
  or equivalently, for any positive measure set $A$ with $T^n(A) \in \mathcal{B}$ ($n \geq 0$),
  \[
  \mu(T^n(A)) \to 1 \ (n \to \infty).
  \]

- **maximal entropy measure**: the measure attaining the maximum of
  \[
  \sup \{ h_{\mu} : \mu \text{ invariant} \}.
  \]

- **intrinsic ergodicity**: the maximal entropy measure is unique.
II. General piecewise monotone transformation

\[ T : [0, 1] \rightarrow [0, 1]. \]

- finite partition of \([0, 1] : P = \{P_1, \ldots, P_N\}.\)
- on each \(P_i\), \(T\) is monotonic, Lipschitz continuous and \(|T'| \geq \rho > 1.\)

Lasota-Yorke 1974: There is an invariant measure \(d\mu = hd\lambda\), where \(d\lambda\) is the Lebesgue measure and \(h\) is a density of bounded variation.

Keller 1978: The set \(\{h \neq 0\}\) is a finite union of intervals.

Wagner 1979: We can decompose \([0, 1] = \bigcup_{i=1}^{s} A_i \cup B\), such that

- on each \(A_i\) there is an invariant measure which is equivalent to the Lebesgue measure restricted to \(A_i\)
- each \(A_i\) can be decomposed as \(A_i = \bigcup_{j=1}^{m_i} A_{ij}\) and \(T^{m_i}\) is exact on each \(A_{ij}\).
- the set \(B\) satisfies \(T^{-1}B \subset B\) and \(\lim_{n \to \infty} \lambda(T^{-n}B) = 0.\)

Hofbauer 1981: The number of maximal entropy measures is finite. If \(T\) is topological transitive, then it is intrinsic ergodic.
III. Dynamical properties - the \((-\beta)\) case

**Ito-Sadahiro 2009** : Let \(h_{-\beta}\) be a real-valued function defined on \((0, 1]\) by

\[
h_{-\beta}(x) = \sum_{n \geq 1, \ T_{-\beta}^n(1) \geq x} \frac{1}{(-\beta)^n}.
\]

Then the measure \(h_{-\beta}(x)d\lambda\) is an invariant measure of \(T_{-\beta}\).

**Remark** : The density may be zero on some intervals. So the invariant measure is not equivalent to the Lebesgue measure. (**Different to the \(\beta\) case**).

**Góra 2007** : for \(\beta > \gamma_1\) (the smallest Pisot number), the transformation \(T_{-\beta}\) is exact and he conjectured that this would hold for all \(\beta > 1\).

**Faller 2008** : \(\beta > 3\sqrt{2}\), \(T_{-\beta}\) admits a unique maximal measure.
IV. Our results-Notations

For each $n \geq 0$, let $\gamma_n$ be the positive real number defined by

$$\gamma_n^{g_n+1} = \gamma_n + 1,$$

with $g_n = \lfloor 2^{n+2}/3 \rfloor$.

Then

$$2 > \gamma_0 > \gamma_1 > \gamma_2 > \cdots > 1.$$

Note that $\gamma_0$ is the golden ratio and that $\gamma_1$ is the smallest Pisot number.

For each $n \geq 0$ and $1 < \beta < \gamma_n$, set

$$G_n(\beta) = \{ G_{m,k}(\beta) | 0 \leq m \leq n, 0 \leq k < g'_{m-1} \},$$

with $g'_{m} = \lfloor 2^{m+2}/3 + 1 \rfloor$.

with open intervals

$$G_{m,k}(\beta) = \begin{cases} (T_{-\beta}^{2^{m+1}+k}(1), T_{-\beta}^{g'_{m}+k}(1)) & \text{if } k \text{ is even,} \\ (T_{-\beta}^{g'_{m}+k}(1), T_{-\beta}^{2^{m+1}+k}(1)) & \text{if } k \text{ is odd.} \end{cases}$$
IV. Our results-Theorems
We call an interval a gap if the density of the invariant measure is zero on it.

Theorem (Liao-Steiner 2010)

If \( \beta \geq \gamma_0 \), then there is no gap. If \( \gamma_{n+1} \leq \beta < \gamma_n \), \( n \geq 0 \), then the set of gaps is \( G_n(\beta) \) which consists of \( g_n = \lfloor 2^{n+2}/3 \rfloor \) disjoint non-empty intervals.

Define

\[
G(\beta) = \begin{cases} 
\emptyset & \text{if } \beta \geq \gamma_0 \ , \\
\bigcup_{I \in G_n(\beta)} I & \text{if } \gamma_{n+1} \leq \beta < \gamma_n \ , \ n \geq 0 .
\end{cases}
\]

Theorem (Liao-Steiner 2010)

The transformation \( T_{-\beta} \) is topological mixing on \( (0, 1] \setminus G(\beta) \),

\[
T_{-\beta}^{-1}(G(\beta)) \subset G(\beta) \quad \text{and} \quad \lim_{n \to \infty} \lambda(T_{-\beta}^{-n}(G(\beta))) = 0 .
\]
V. Our results-Theorems (continue)

Define a morphism on the symbolic space \(\{1, 2\}^\mathbb{N}\) by

\[\varphi : 1 \mapsto 2, \quad 2 \mapsto 211.\]

**Theorem (Liao-Steiner 2010)**

*For every* \(n \geq 0\), *we have*

\[d_{-\gamma_n}(1) = \varphi^n(21^\omega).\]

*Hence*

\[
\lim_{\beta \to 1} d_{-\beta}(1) = \lim_{n \to \infty} \varphi^n(2) = 211222112112112122 \cdots .
\]

**Remark:** \(g'_m = |\varphi^m(2)| = |\varphi^{m+1}(1)|\) and \(2^{m+1} = |\varphi^m(21)|.\)
VI. Our results-Corollaries

**Corollary**

*For any \( \beta > 1 \), the transformation \( T_{-\beta} \) is exact.*

**Corollary**

*For any \( \beta > 1 \), the transformation \( T_{-\beta} \) has a unique maximal measure, and hence is intrinsic ergodic.*

**Proofs**: From the topological mixing: for each interval \( J \subset (0, 1] \setminus G(\beta) \), there is \( n \geq 1 \) such that \( f^n(J) = (0, 1] \setminus G(\beta) \).
VII. Our results-Proofs
For every word \( a_1 \cdots a_n \in \{1, 2\}^n \), \( n \geq 0 \), define the polynomial

\[
P_{a_1 \cdots a_n} = (-X)^n + \sum_{k=1}^{n} a_k (-X)^{n-k} \in \mathbb{Z}[X]
\]

Lemma

For \( 1 \leq m < n \), we have

\[
P_{a_1 \cdots a_n} = (-X)^{n-m} (P_{a_1 \cdots a_m} - 1) + P_{a_{m+1} \cdots a_n}.
\]

For \( n \geq 0 \) we have the identities:

1. \( X^{\frac{1+(-1)^n}{2}} P_{\varphi^n(2)} + X^{\frac{1-(-1)^n}{2}} P_{\varphi^n(11)} = X + 1 = X^{\frac{1+(-1)^n}{2}} + X^{\frac{1-(-1)^n}{2}} \)

2. \( 1 - P_{\varphi^n(1)} = X^{\frac{1+(-1)^n}{2}} \prod_{k=0}^{n-1} \left(X^{\varphi^k(1)} - 1\right) \)

3. \( P_{\varphi^n(21)} - P_{\varphi^n(2)} = (X^{g_n+1} - X - 1) \prod_{k=0}^{n-1} \left(X^{\varphi^k(1)} - 1\right) \)
Thank you for your attention!