

A Residue Approach to the Finite Field Arithmetics

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Introduction to Residue Systems

Introduction to Residue Systems

- ▶ In some applications, like **cryptology**, we use finite field arithmetics on huge numbers or large polynomials.
- ▶ **Residue systems** are a way to **distribute the calculus** on small arithmetic units.
- ▶ Are these systems suitable for **finite field arithmetics**?

Residue Number Systems in \mathbb{F}_p , p prime

- ▶ Modular arithmetic mod p , elements are considered as integers.
- ▶ Residue Number System
 - ▶ RNS base: a set of coprime numbers (m_1, \dots, m_k)
 - ▶ RNS representation: (a_1, \dots, a_k) with $a_i = |A|_{m_i}$
 - ▶ Full parallel operations mod M with $M = \prod_{i=1}^k m_i$
 $(|a_1 \otimes b_1|_{m_1}, \dots, |a_n \otimes b_n|_{m_n}) \rightarrow A \otimes B \pmod{M}$
- ▶ Very fast product, but an extension of the base could be necessary and a reduction modulo p is needed.

Residue Number Systems in \mathbb{F}_p , p prime

- ▶ $\Phi(m) = \sum_{\substack{p \leq m \\ p \text{ prime}}} \log p = \log \prod_{\substack{p \leq m \\ p \text{ prime}}} p \sim m$
- ▶ If $2^{m-1} \leq M < 2^m$ then the size moduli is of order $\mathcal{O}(\log m)$.
- ▶ In other words, if addition and multiplication have complexities of order $\Theta(f(m))$ then in RNS the complexities become $\Theta(f(\log m))$.

Lagrange representations in \mathbb{F}_{p^k} with $p > 2k$

- ▶ **Arithmetic modulo $I(X)$** , an irreducible \mathbb{F}_p polynomial of degree k . Elements of \mathbb{F}_{p^k} are considered as \mathbb{F}_p polynomials of degree lower than k .
- ▶ **Lagrange representation**
 - ▶ is defined by k **different** points e_1, \dots, e_k in \mathbb{F}_p . ($k \leq p$.)
 - ▶ A polynomial $A(X) = \alpha_0 + \alpha_1 X + \dots + \alpha_{k-1} X^{k-1}$ over \mathbb{F}_p is given in Lagrange representation by:

$$(a_1 = A(e_1), \dots, a_k = A(e_k)).$$

- ▶ **Remark:** $a_j = A(e_j) = A(X) \bmod (X - e_j)$. If we note $m_j(X) = (X - e_j)$, we obtain a similar representation as RNS.
- ▶ **Operations are made independently on each $A(e_j)$** (like in FFT or Tom-Cook approaches). We need to extend to $2k$ points for the product.



Trinomial residue in \mathbb{F}_2^n

- ▶ Arithmetic modulo $I(X)$, an irreducible \mathbb{F}_2 polynomial of degree n . Elements of \mathbb{F}_2^n are considered as \mathbb{F}_2 polynomials of degree lower than n .
- ▶ Trinomial representation
 - ▶ is defined by a set of k coprime trinomials $m_i(X) = X^d + X^{t_i} + 1$, with $k \times d \geq n$,
 - ▶ an element $A(X)$ is represented by $(a_1(X), \dots, a_k(X))$ with $a_i(X) = A(X) \bmod m_i(X)$.
 - ▶ This representation is equivalent to RNS.
- ▶ Operations are made independently for each $m_i(X)$

Residue Systems

- ▶ Residue systems could be an issue for computing efficiently the product.
- ▶ The main operation is now the modular reduction for constructing the finite field elements.
- ▶ The choice of the residue system base is important, it gives the complexity of the basic operations.

Modular reduction in Residue Systems

Reduction of Montgomery on \mathbb{F}_p

- ▶ The most used reduction algorithm is due to Montgomery (1985)[9]
- ▶ For reducing A modulo p ,
 one evaluates $q = -(Ap^{-1}) \bmod 2^s$,
 then one constructs $R = (A + qp)/2^s$.
 The obtained value satisfies: $R \equiv A \times 2^{-s} \pmod{p}$ and
 $R < 2p$ if $A < p2^s$.
 We note $\text{Montg}(A, 2^s, p) = R$.
- ▶ **Montgomery notation:** $A' = A \times 2^s \bmod p$
 $\text{Montg}(A' \times B', 2^s, p) \equiv (A \times B) \times 2^s \pmod{p}$

Residue version of Montgomery Reduction

- ▶ The residue base is such that $p < M$
(or $\deg M(X) \geq \deg I(X)$)
- ▶ We use an auxiliary base such that $p < M'$
(or $\deg M'(X) \geq \deg I(X)$), M' and M coprime.
(Exact product, and existence of M^{-1})
- ▶ Steps of the algorithm
 1. $Q = -(Ap^{-1}) \bmod M$ (calculus in base M)
 2. Extension of the representation of Q to the base M'
 3. $R = (A + Qp) \times M^{-1}$ (calculus in base M')
 4. Extension of the representation of R to the base M
- ▶ The values are represented in the two bases.

Extension of Residue System Bases (from M to M')

The extension comes from the Lagrange interpolation.

If (a_1, \dots, a_k) is the residue representation in the base M , then

$$A = \sum_{i=1}^k a_i \times \left[\frac{M}{m_i} \right]_{m_i}^{-1} \times \frac{M}{m_i} - \alpha M$$

The factor α can be, in certain cases, neglected or computed. [1]

Another approach consists in the Newton interpolation where A is correctly reconstructed. [4]

In the polynomial case, the term $-\alpha M$ vanishes.

Extension for Q

By the CRT

$$\hat{Q} = \sum_{i=1}^n \left| q_i |M_i|_{m_i}^{-1} \right|_{m_i} M_i = Q + \alpha M$$

where $0 \leq \alpha < n$.

When \hat{Q} has been computed it is possible to compute \hat{R} as

$$\begin{aligned} \hat{R} &= (AB + \hat{Q}p)M^{-1} = (AB + Qp + \alpha Mp)M^{-1} \\ &= (AB + Qp)M^{-1} + \alpha p \end{aligned}$$

so that $\hat{R} \equiv R \equiv ABM^{-1} \pmod{p}$, which is sufficient for our purpose. Also, assuming that $AB < pM$ we find that $\hat{R} < (n+2)p$ since $\alpha < n$.

Extension R

Shenoy et Kumaresan (1989):

$$\text{we have } \left(\sum_{i=1}^n M_i \left| \left| M_i \right|_{m_i}^{-1} r_i \right|_{m_i} \right) = R + \alpha \times M$$

$$\alpha = \left| \left| M \right|_{m_{n+1}}^{-1} \left(\sum_{i=1}^n \left| M_i \right|_{m_i} \left| \left| M_i \right|_{m_i}^{-1} r_i \right|_{m_i} \right) \right|_{m_{n+1}} - \left| R \right|_{m_{n+1}} \right|_{m_{n+1}}$$

$$\tilde{r}_j = \left| \sum_{i=1}^n \left| M_i \right|_{m_i} \left| \left| M_i \right|_{m_i}^{-1} r_i \right|_{m_i} \right|_{\tilde{m}_j} - \left| \alpha M \right|_{\tilde{m}_j} \right|_{\tilde{m}_j}$$

Extension of Residue System Bases

We first translate in an intermediate representation (MRS):

$$\left\{ \begin{array}{l} \zeta_1 = a_1 \\ \zeta_2 = (a_2 - \zeta_1) m_1^{-1} \bmod m_2 \\ \zeta_3 = ((a_3 - \zeta_1) m_1^{-1} - \zeta_2) m_2^{-1} \bmod m_3 \\ \vdots \\ \zeta_n = (\dots ((a_n - \zeta_1) m_1^{-1} - \zeta_2) m_2^{-1} - \dots - \zeta_{n-1}) m_{n-1}^{-1} \bmod m_n. \end{array} \right.$$

We evaluate A , with Horner's rule, as

$$A = (\dots ((\zeta_n m_{n-1} + \zeta_{n-1}) m_{n-2} + \dots + \zeta_3) m_2 + \zeta_2) m_1 + \zeta_1.$$

Features of the residue system

- ▶ Efficient multiplication, the cost being the cost of one multiplication on one residue.
- ▶ Costly reduction: $O(k^{1.6})$ for trinomials [4], $2k^2 + 3k$ for RNS [1], $O(k^2)$ for Lagrange representation [5].
- ▶ If we take into account that most of the operations are multiplications by a constant, the cost can be considerably smaller.

Applications to Cryptography

Elliptic curve cryptography

- ▶ The main idea comes from the **efficiency of the product and the cost of the reduction in Residue Systems**.
- ▶ We try to minimize the number of reductions. A reduction is not necessary after each operation. Clearly, **for a formula like $A \times B + C \times D$, only one reduction is needed**.
- ▶ Elliptic Curve Cryptography is based on points addition. We use appropriate forms (Hessian, Jacobi, Montgomery...) and coordinates: projective, Jacobian or Chudnowski...
- ▶ **For 512 bits values Residues Systems for curves defined over a prime field, are more efficient than classical representations.[2]**

Pairings

- ▶ To summarize we define a pairing as follows: let G_1 and G_2 be two additive abelian groups of cardinal n and G_3 a multiplicative group of cardinal n .
- ▶ A pairing is a function $e : G_1 \times G_2 \rightarrow G_3$ which verifies the following properties: Bilinearity, Non-degeneracy.
- ▶ For pairings defined on an elliptic curve E over a finite field \mathbb{F}_p , we have $G_1 \subset E(\mathbb{F}_p)$, $G_2 \subset E(\mathbb{F}_{p^k})$ and $G_3 \subset \mathbb{F}_{p^k}$, where k is the smallest integer such that n divides $p^k - 1$, k is called the embedded degree of the curve.




Pairings






- ▶ The construction of the pairing involves values over \mathbb{F}_p and \mathbb{F}_{p^k} into the formulas. An approach with Residue Systems, similar to the one made on ECC could be interesting.[3]
- ▶ k is most of the time chosen as a small power of 2 and 3 for algorithmic reasons. Residue arithmetics allow to pass over this restriction.
- ▶ With pairings, we can also imagine two levels of Residue Systems: one over \mathbb{F}_p and one over \mathbb{F}_{p^k} .




Conclusion on Residue Systems

Conclusion

- ▶ If your number system is not efficient, then it remains to try the residues.

-  Bajard, J.C., Didier, L.S., Kornerup, P.: Modular multiplication and base extension in residue number systems. 15th IEEE Symposium on Computer Arithmetic, 2001 Vail Colorado USA pp. 59–65
-  Bajard, J.C., Duquesne, S., Ercegovic M. and Meloni N.: Residue systems efficiency for modular products summation: Application to Elliptic Curves Cryptography, in Advanced Signal Processing Algorithms, Architectures, and Implementations XVI, SPIE 2006, San Diego, USA.
-  Bajard, J.C. and ElMrabet N.: Pairing in cryptography: an arithmetic point of view, Advanced Signal Processing Algorithms, Architectures, and Implementations XVII, part of the SPIE Optics & Photonics 2007 Symposium. August 2007 San Diego, USA.

-  J.C. Bajard, L. Imbert, and G. A. Jullien: Parallel Montgomery Multiplication in $GF(2^k)$ using Trinomial Residue Arithmetic, 17th IEEE symposium on Computer Arithmetic, 2005, Cape Cod, MA, USA.pp. 164-171
-  J.C. Bajard, L. Imbert et Ch. Negre, Arithmetic Operations in Finite Fields of Medium Prime Characteristic Using the Lagrange Representation, journal IEEE Transactions on Computers, September 2006 (Vol. 55, No. 9) p p. 1167-1177
-  Bajard, J.C., Meloni, N., Plantard, T.: Efficient RNS bases for Cryptography IMACS'05, Applied Mathematics and Simulation, (2005)
-  Garner, H.L.: The residue number system. IRE Transactions on Electronic Computers, EL **8:6** (1959) 140–147
-  Knuth, D.: Seminumerical Algorithms. The Art of Computer Programming, vol. 2. Addison-Wesley (1981)

-  Montgomery, P.L.: Modular multiplication without trial division. *Math. Comp.* **44:170** (1985) 519–521
-  Svoboda, A. and Valach, M.: *Operational Circuits. Stroje na Zpracovani Informaci, Sbornik III, Nakl. CSAV, Prague, 1955, pp.247-295.*
-  Szabo, N.S., Tanaka, R.I.: *Residue Arithmetic and its Applications to Computer Technology.* McGraw-Hill (1967)