Extensions naturelles des bêta-transformations généralisées et pavages

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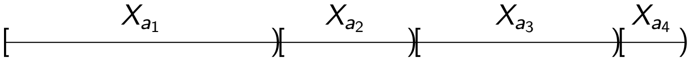
(travail en commun avec Charlene Kalle, Universiteit Utrecht, en ce moment à l'University of Warwick)

19 Janvier 2010
Séminaire de Probabilités et Théorie Ergodique, LAMFA, Amiens

Transformations generating digital expansions in base β

- Let $\beta > 1$ be a real number.
- Let $A \subset \mathbb{R}$ be a finite set, the digit set.
- Let $X = \bigcup_{a \in A} X_a$, where the X_a's are bounded intervals (or finite unions of them) and the union is disjoint, typically



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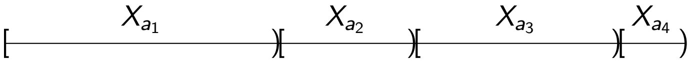

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$$T(x) = \beta x - a \quad \text{if} \quad x \in X_a \, .$$

  Assume that $T(X) = X$.
- Define the digit sequence $b(x) = b_1(x) b_2(x) \cdots$ by

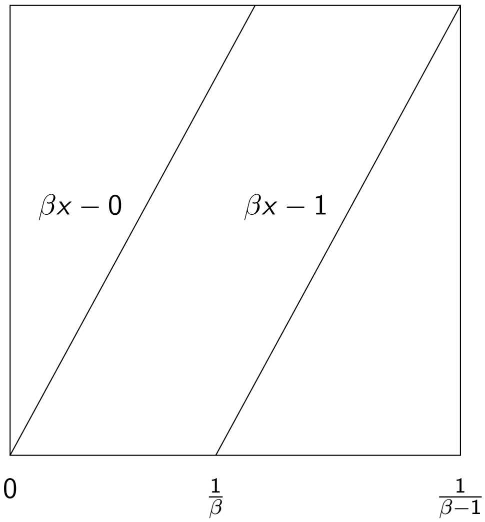
$$b_n(x) = a \quad \text{if} \quad T^{n-1}(x) \in X_a \, .$$

Then

$$x = \frac{b_1(x)}{\beta} + \frac{T(x)}{\beta} = \frac{b_1(x)}{\beta} + \frac{b_2(x)}{\beta^2} + \frac{T^2(x)}{\beta^2} = \cdots = \sum_{n=1}^{\infty} \frac{b_n(x)}{\beta^n} \, ,$$

and we call $b(x)$ the T-expansion of x.

Example: $1 < \beta \leq 2$, $A = \{0, 1\}$



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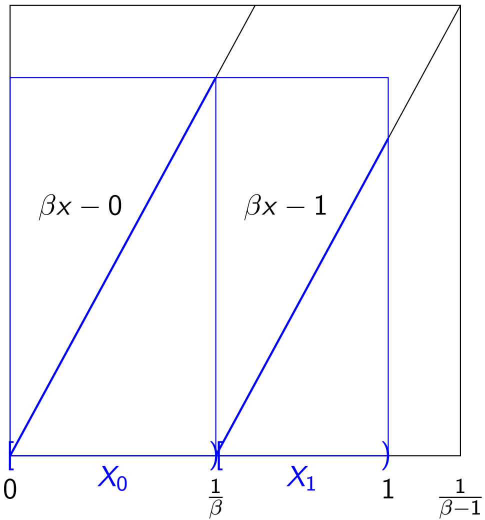

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$X_1 = [1/\beta, 1)$

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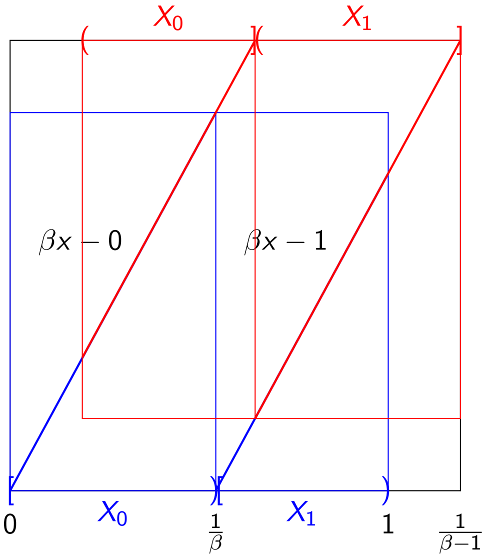

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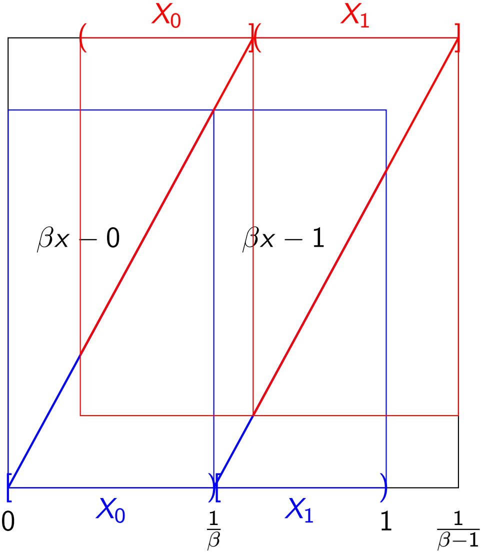

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intermediate transformations
different alphabets

Natural extensions

Consider the dynamical system $(X, \mathcal{B}, T)$, where $\mathcal{B}$ is the Lebesgue σ-algebra on X. The map T is not invertible.

A natural extension of $(X, \mathcal{B}, T)$ is an invertible system $(\widehat{X}, \widehat{\mathcal{B}}, \widehat{T})$ such that:

- There is a surjective and measurable map $\pi : \widehat{X} \to X$ with

$$\pi \circ \widehat{T} = T \circ \pi \,.$$

- This system is the smallest in the sense of σ-algebras:

$$\bigvee_{n \geq 0} \widehat{T}^n(\pi^{-1}(\mathcal{B})) = \widehat{\mathcal{B}} \,.$$

Conditions on β, companion matrix, eigenvectors

Let $\beta > 1$ be a Pisot unit, i.e., an algebraic integer with $|\beta_j| < 1$ for every Galois conjugate $\beta_j \neq \beta$ of β and minimal polynomial $X^d - c_1 X^{d-1} - c_2 X^{d-2} - \cdots - c_d \in \mathbb{Z}[X]$ with $c_d \in \{-1, 1\}$.

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Let M be the companion matrix

$$M = \begin{pmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad |\det M| = |c_d| = 1\,.$$

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For $1 \le j \le d$, let $\mathbf{v}_j = \nu_j(\beta_j^{d-1}, \ldots, \beta_j, 1)^t$, $\nu_j \in \mathbb{C}$, be a right eigenvector of M to the eigenvalue β_j (with $\beta_1 = \beta$), such that $\sum_{j=1}^d \mathbf{v}_j = \mathbf{e}_1 = (1, 0, \ldots, 0)^t$.

Let H be the hyperplane of $\mathbb{R}^d$ spanned by the real and imaginary parts of $\mathbf{v}_2, \ldots, \mathbf{v}_d$, then M is contractive on H.

Let $\mathbf{e}_1 = \mathbf{e}_\beta + \mathbf{e}_H$ with $\mathbf{e}_\beta = \mathbf{v}_1$ ($M\mathbf{e}_\beta = \beta\,\mathbf{e}_\beta$), $\mathbf{e}_H = \sum_{j=2}^d \mathbf{v}_j \in H$.

Natural extension domain

Let $\mathcal{S}$ denote the set of two-sided sequences $u = (u_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}}$ such that every suffix of u is a T-expansion of some $x \in X$:

$$\mathcal{S} = \{u \in A^{\mathbb{Z}} \mid u_n u_{n+1} \cdots \in b(X) \text{ for all } n \in \mathbb{Z}\}.$$

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Assume that $A \subset \mathbb{Z}$ (for simplicity; with small modifications, everything holds for $A \subset \mathbb{Q}(\beta)$). Define $\psi : \mathcal{S} \to \mathbb{R}^d$ by

$$\psi(u) = \sum_{n\geq 1} u_n\, \beta^{-n}\mathbf{e}_\beta - \sum_{n\leq 0} u_n\, M^{-n}\mathbf{e}_H\,.$$

Let $\widehat{X} = \psi(\mathcal{S}) = \bigcup_{a\in A} \widehat{X}_a$ be the natural extension domain, with

$$\widehat{X}_a = \left\{\psi(u) \mid u \in \mathcal{S},\ u_1 = a\right\}.$$

Note that the union is disjoint since the $\mathbf{e}_\beta$-coordinates in $\widehat{X}_a$ are different from those in $\widehat{X}_{a'}$ for all $a \neq a'$, and that $\widehat{X}$ is bounded.

Natural extension transformation

Define the natural extension transformation $\widehat{T} : \widehat{X} \to \widehat{X}$ by

$$\widehat{T}(\mathbf{x}) = M\mathbf{x} - a\,\mathbf{e}_1 \quad \text{for } \mathbf{x} \in \widehat{X}_a \,.$$

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Write $\mathbf{x} = x\,\mathbf{e}_\beta + \mathbf{y}$ with $x \in X$, $\mathbf{y} \in H$, then

$$\widehat{T}(\mathbf{x}) = \widehat{T}(x\,\mathbf{e}_\beta + \mathbf{y}) = \underbrace{(\beta x - a)}_{T(x)}\mathbf{e}_\beta + M\mathbf{y} - a\,\mathbf{e}_H\,.$$

Let π be the projection on the $\mathbf{e}_\beta$-coordinate, then $\pi \circ \widehat{T} = T \circ \pi$.

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Since $\psi(u) = \sum_{n\geq 1} u_n\,\beta^{-n}\mathbf{e}_\beta - \sum_{n\leq 0} u_n\,M^{-n}\mathbf{e}_H$, we have

$$\widehat{T}\big(\psi(u)\big) = \sum_{n\geq 1} u_{n+1}\beta^{-n}\mathbf{e}_\beta - \sum_{n\leq 0} u_{n+1}\,M^{-n}\mathbf{e}_H = \psi\big(\sigma(u)\big) \in \widehat{X},$$

where σ denotes the left-shift on $\mathcal{S}$, and

$$\widehat{T}\big(\widehat{X}\big) = \widehat{T}\big(\psi(\mathcal{S})\big) = \psi\big(\sigma(\mathcal{S})\big) = \psi(\mathcal{S}) = \widehat{X}\,.$$

Natural extension

Let λ_d denote the Lebesgue measure on $\mathbb{R}^d$, then $|\det M| = 1$ yields

$$\lambda_d\big(\widehat{T}(\widehat{X}_a)\big) = \lambda_d\big(M\widehat{X}_a - a\,\mathbf{e}_1\big) = \lambda_d\big(\widehat{X}_a\big)\,.$$

Recall that $\widehat{X} = \bigcup_{a\in A}\widehat{X}_a$ (disjoint) and $\widehat{T}\big(\widehat{X}\big) = \widehat{X}$, hence

$$\lambda_d\big(\widehat{T}(\widehat{X}_a) \cap \widehat{T}(\widehat{X}_{a'})\big) = 0 \quad \text{for } a \neq a'\,.$$

Therefore, $\widehat{T}$ is bijective, up to a set of measure zero.

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It can be shown that $\mathbb{Q}^d \subset \bigcup_{\mathbf{z}\in\mathbb{Z}^d} \big(\mathbf{z} + \widehat{X}\big)$, and that $\lambda_d\big(\widehat{X}\big) = \lambda_d\big(\mathrm{closure}(\widehat{X})\big)$. Since $\widehat{X}$ is bounded, $0 < \lambda_d\big(\widehat{X}\big) < \infty$.

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$\big(\widehat{X}, \widehat{\mathcal{B}}, \widehat{T}\big)$ is a natural extension of $(X, \mathcal{B}, T)$, up to measure zero.

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Theorem

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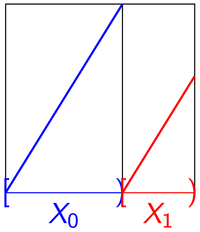
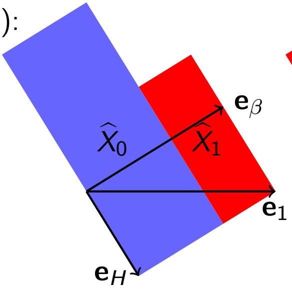
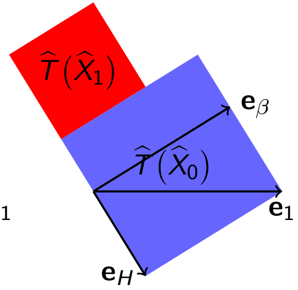
Since $\widehat{T}$ preserves the Lebesgue measure, an ACIM of T is given by the projection of the Lebesgue measure of $\widehat{X}$ on the $\mathbf{e}_\beta$-coordinate.

Examples of natural extensions for the β-transformation

$\beta^2 = \beta + 1$ ($\beta \approx 1.618$):







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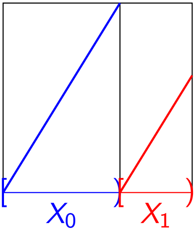
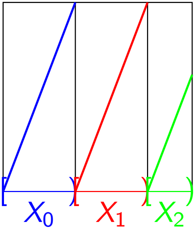
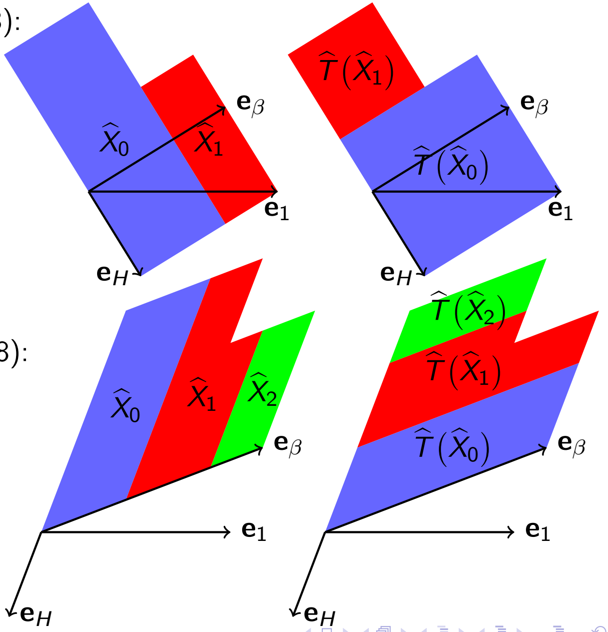


$\beta^2 = 3\beta - 1$ ($\beta \approx 2.618$):





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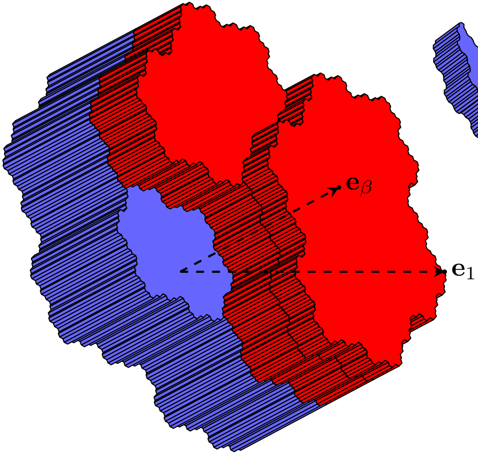
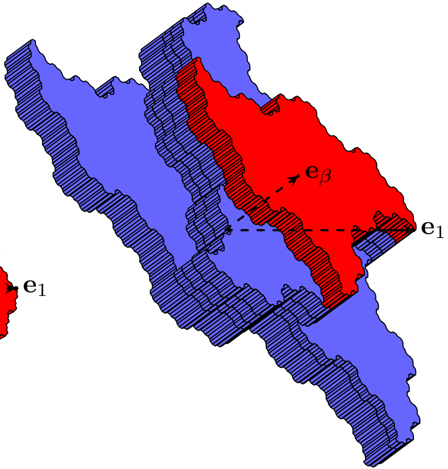
$\beta^3 = \beta^2 + \beta + 1$ ($\beta \approx 1.8393$)
(Tribonacci number)



$\beta^3 = \beta + 1$ ($\beta \approx 1.3247$)
(smallest Pisot number)



Natural extensions for minimal weight transformations

If β is a Pisot unit and $A \subset \mathbb{Z}$, then let the weight of T be

$$w(T) = \frac{1}{\lambda_d\big(\widehat{X}\big)} \sum_{a \in A} |a| \, \lambda_d\big(\widehat{X}_a\big) \, .$$

T is a minimal weight transformation if $w(T) \leq w(S)$ for every S with the same β.

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T is a minimal weight transformation if $w(T) \leq w(S)$ for every S with the same β.

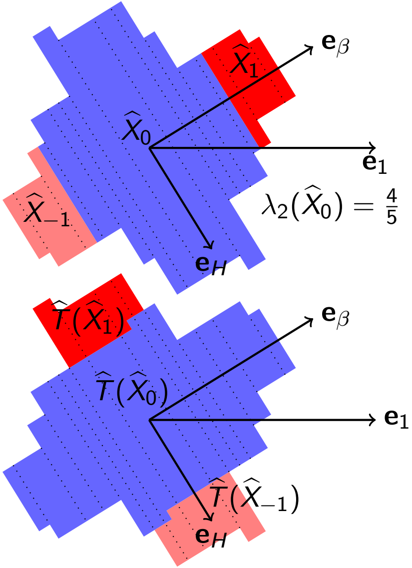
Theorem (Frougny–St 2008)

Let T be defined by $\beta > 1$ and $A = \{-1, 0, 1\}$, $\frac{1}{2} \leq \alpha \leq \frac{1}{\beta-1}$,
$X_{-1} = [-\alpha, -\alpha/\beta)$, $X_0 = [-\alpha/\beta, \alpha/\beta)$, $X_1 = [\alpha/\beta, \alpha)$.

If $\beta^2 = \beta + 1$ and $\frac{\beta^2}{\beta^2+1} \leq \alpha \leq \frac{2\beta}{\beta^2+1}$,
or $\beta^3 = \beta^2 + \beta + 1$ and $\frac{\beta}{\beta+1} \leq \alpha \leq \frac{2+1/\beta}{\beta+1}$,
or $\beta^3 = \beta + 1$ and $\frac{\beta^3}{\beta^2+1} \leq \alpha \leq \frac{\beta^2+1/\beta}{\beta^2+1}$,
then T is a minimal weight transformation.

Examples of minimal weight transformations

$\beta = \frac{1+\sqrt{5}}{2}$, $\alpha = \frac{\beta^2+\beta^{-3}}{\beta^2+1} \approx 0.7889$



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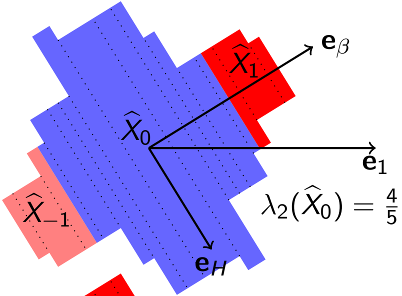
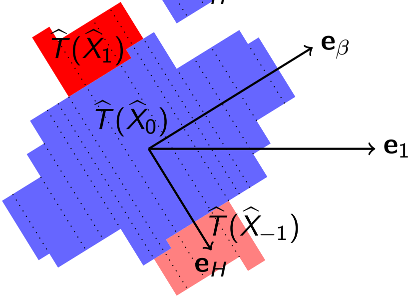
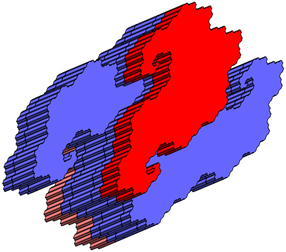
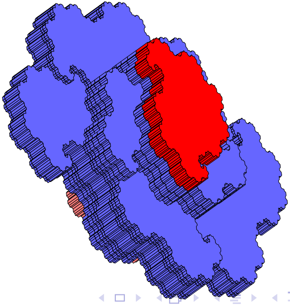




$\beta^3 = \beta^2 + \beta + 1$, $\alpha = \frac{\beta}{\beta+1} \approx 0.6478$



$\beta^3 = \beta + 1$, $\alpha = \frac{\beta^3}{\beta^2+1} \approx 0.8439$



Structure of $\widehat{X}$

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Assume $A = \{a_1, \ldots, a_k\}$, $X_{a_j} = [\delta_{j-1}, \delta_j)$, $\delta_0 < \delta_1 < \cdots < \delta_k$.

Define $\widetilde{X}_{a_j} = (\delta_{j-1}, \delta_j]$, $\widetilde{X} = \bigcup_{a\in A} \widetilde{X}_a$, $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ with

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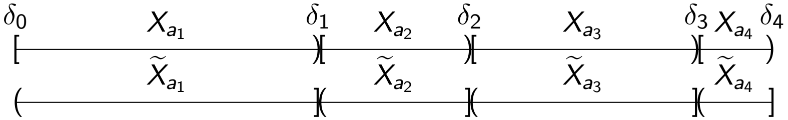

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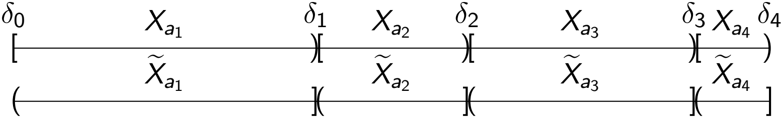


Let $m_j > 0$ be minimal such that $T^{m_j}(\delta_j) = \widetilde{T}^{m_j}(\delta_j)$, $0 < j < k$, with $m_j = \infty$ if $T^n(\delta_j) \neq \widetilde{T}^n(\delta_j)$ for all $n > 0$, and set

$$\mathcal{V} = \{\delta_0\} \cup \bigcup_{0<j<k} \bigcup_{0<n<m_j} \{T^n(\delta_j)\,,\ \widetilde{T}^n(\delta_j)\} \setminus \{\delta_k\}\,.$$

Proposition

If $x, y \in X$, $x < y$ and $(x, y] \cap \mathcal{V} = \emptyset$, then $\mathcal{D}_x = \mathcal{D}_y$.

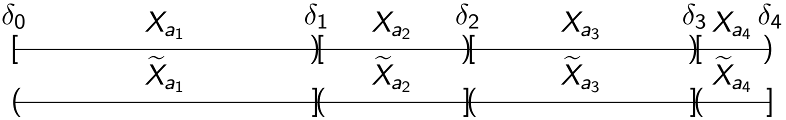
Structure of $\widehat{X}$

Let $\mathcal{D}_x = \{\mathbf{y} \mid x\,\mathbf{e}_\beta + \mathbf{y} \in \widehat{X}\}$ be the x-fiber of $\widehat{X}$. $\mathcal{D}_x$ is compact.

Assume $A = \{a_1, \ldots, a_k\}$, $X_{a_j} = [\delta_{j-1}, \delta_j)$, $\delta_0 < \delta_1 < \cdots < \delta_k$.

Define $\widetilde{X}_{a_j} = (\delta_{j-1}, \delta_j]$, $\widetilde{X} = \bigcup_{a \in A} \widetilde{X}_a$, $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ with

$$\widetilde{T}(x) = \beta x - a \quad \text{if} \quad x \in \widetilde{X}_a\,.$$



Let $m_j > 0$ be minimal such that $T^{m_j}(\delta_j) = \widetilde{T}^{m_j}(\delta_j)$, $0 < j < k$, with $m_j = \infty$ if $T^n(\delta_j) \neq \widetilde{T}^n(\delta_j)$ for all $n > 0$, and set

$$\mathcal{V} = \{\delta_0\} \cup \bigcup_{0<j<k} \bigcup_{0<n<m_j} \{T^n(\delta_j)\,,\ \widetilde{T}^n(\delta_j)\} \setminus \{\delta_k\}\,.$$

Proposition

If $x, y \in X$, $x < y$ and $(x, y] \cap \mathcal{V} = \emptyset$, then $\mathcal{D}_x = \mathcal{D}_y$.

$\mathcal{V}$ is finite if and only if $m_j < \infty$ or the T- and $\widetilde{T}$-orbits of δ_j are eventually periodic, $0 < j < k$.

Periodic T-expansions

Theorem

The T-expansion (and $\widetilde{T}$-expansion) of x is eventually periodic if and only if $x \in \mathbb{Q}(\beta) \cap X$.

(Here, β can be a Pisot number, not necessarily a Pisot unit.)
cf. Bertrand (1977), K. Schmidt (1980), Frank–Robinson (2008)

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For $x \in \mathbb{Q}(\beta)$, let $x^{(j)}$ be the image of x by the automorphism from $\mathbb{Q}(\beta)$ to $\mathbb{Q}(\beta_j)$ mapping β to β_j, and

$$\Psi : \mathbb{Q}(\beta) \to \mathbb{Q}^d, \quad x \mapsto \sum_{j=1}^{d} x^{(j)} \mathbf{v}_j \,,$$

in particular $\Psi(x) = x\,\mathbf{e}_1$ if $x \in \mathbb{Q}$.

Theorem

The T-expansion of x is purely periodic if and only if $x \in \mathbb{Q}(\beta) \cap X$ and $\Psi(x) \in \widehat{X}$.

cf. Ito–Rao (2006), Berthé–Siegel (2005)

(Multiple) Tilings of H

Let $$\Phi : \mathbb{Q}(\beta) \to H, \quad x \mapsto \sum_{j=2}^{d} x^{(j)} \mathbf{v}_j = \Psi(x) - x\mathbf{e}_\beta,$$

For $x \in \mathbb{Z}[\beta] \cap X$, set

$$\mathcal{T}_x = \operatorname*{Lim}_{n\to\infty} \Phi\big(\beta^n T^{-n}(x)\big) = \operatorname*{Lim}_{n\to\infty} M^n \Phi\big(T^{-n}(x)\big) = \Phi(x) - \mathcal{D}_x,$$

where Lim denotes the Hausdorff limit.

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where Lim denotes the Hausdorff limit.

Every tile $\mathcal{T}_x$ subdivides into contracted copies of other tiles:

$$\mathcal{T}_x = \bigcup_{y \in T^{-1}(x)} M\,\mathcal{T}_y\,, \quad \mathcal{D}_x = \bigcup_{y \in T^{-1}(x)} \Big(M\,\mathcal{D}_y - \Phi\big(b_1(y)\big)\Big)\,.$$

If $\mathcal{V}$ is finite, this gives a graph-directed iterated function system for $\{\mathcal{D}_x\}_{x\in\mathcal{V}}$, the unions are disjoint up to sets of measure zero, $\lambda_{d-1}(\partial\mathcal{T}_x) = 0$; $\lambda_{d-1}(\mathcal{T}_x) > 0$ iff x is in the support of the ACIM.

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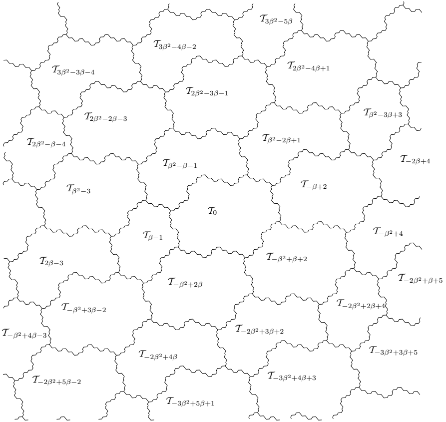
Theorem

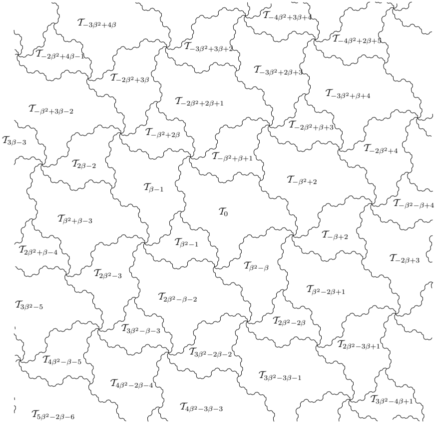
If $\mathcal{V}$ is finite, then $\{\mathcal{T}_x\}_{x\in\mathbb{Z}[\beta]\cap X}$ is a multiple tiling of H. (There exists $m\geq 1$ such that almost every point lies in exactly m tiles.)

cf. Thurston (1989), Praggastis (1999), Akiyama (1999, 2002), Ito–Rao (2006), Berthé–Siegel (2005)

Examples of tilings of H for the β-transformations

$\beta^3 = \beta^2 + \beta + 1$ $\qquad\qquad$ $\beta^3 = \beta + 1$





Conjecture

Let β be a Pisot unit and $T : [0,1) \to [0,1)$, $x \mapsto \beta x - \lfloor \beta x \rfloor$, be the (greedy) β-transformation. Then $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a tiling of H.

Sofic shifts

Assume $A = \{a_1, \ldots, a_k\}$, $X_{a_j} = [\delta_{j-1}, \delta_j)$, $\delta_0 < \delta_1 < \cdots < \delta_k$, and denote the $\widetilde{T}$-expansion of $x \in \widetilde{X}$ by $\tilde{b}(x)$.

Theorem

A sequence $a_{v_1} a_{v_2} \cdots$ is the T-expansion of some $x \in X$ iff

$$b(\delta_{v_n - 1}) \leq_{\mathrm{lex}} a_{v_n} a_{v_{n+1}} \cdots <_{\mathrm{lex}} \tilde{b}(\delta_{v_n}) \quad \textit{for all } n \geq 1\,.$$

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The closure of

$$\mathcal{S} = \{u \in A^{\mathbb{Z}} \mid u_n u_{n+1} \cdots \in b(X) \text{ for all } n \in \mathbb{Z}\}$$

is a sofic shift if and only if $b(\delta_{j-1})$ and $\tilde{b}(\delta_j)$ are eventually periodic for all $1 \leq j \leq k$, i.e., $\delta_j \in \mathbb{Q}(\beta)$.

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$\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ can be a tiling even if the closure of $\mathcal{S}$ is not sofic!

Tiling of $\mathbb{R}^d$ by $\widehat{X}$, partition of $\mathbb{R}^d/\mathbb{Z}^d$ by $\{\widehat{X}_a\}_{a\in A}$

Theorem

$\{\mathcal{T}_x\}_{x\in\mathbb{Z}[\beta]\cap X}$ is a tiling of H (multiple tiling of degree $m = 1$) if and only if $\{\mathbf{z} + \widehat{X}\}_{\mathbf{z}\in\mathbb{Z}^d}$ is a tiling of $\mathbb{R}^d$.

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Theorem

If $\mathbf{0}\in\mathcal{D}_x$, then $\beta^{n-1}x\,\mathbf{e}_\beta \in \widehat{X}_{b_n(x)}$ (mod $\mathbb{Z}^d$).

If $\{\mathbf{z}+\widehat{X}\}_{\mathbf{z}\in\mathbb{Z}^d}$ is a tiling of $\mathbb{R}^d$ and $\mathbf{0}$ is an inner point of $\mathcal{D}_x$, then

$$b_n(x) = a \quad \text{if and only if} \quad \beta^{n-1}x\,\mathbf{e}_\beta \in \widehat{X}_a \pmod{\mathbb{Z}^d}.$$

(This is also true for "most" x if $\mathbf{0}$ is not an inner point of $\mathcal{D}_x$.)

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This can be used e.g. to calculate the statistics of the n-th digit in $\{P(m)\mid m\in\mathbb{Z},\ P(m)\in X\}$ for a polynomial P, cf. St (2002).

Degree of the multiple tiling

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Proposition

Let $z \in \mathbb{Z}[\beta] \cap [0, \infty)$ and $n \geq 0$ such that $\beta^{-n} z \in [0, \varepsilon)$. Then $\Phi(z)$ lies exactly in the tiles $\mathcal{T}_{T^n(x+\beta^{-n}z)}$, $x \in \mathcal{P}$.

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$\Phi\big(\mathbb{Z}[\beta] \cap [0, \infty)\big)$ is dense in H. Consider two properties:

(F) : $\mathcal{P}$ consists only of one element .

(W) : $\exists\, y \in \mathcal{P} : \forall\, x \in \mathcal{P}\ \exists\, z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)\,,\ n \geq 0 :$
$$T^n(x + z) = T^n(y + z) = y\,.$$

Theorem

$\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a tiling ($m = 1$) if and only if (W) holds.

cf. Akiyama (2002); (F) $\Rightarrow$ (W)

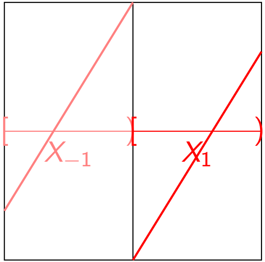
Example of a multiple tiling

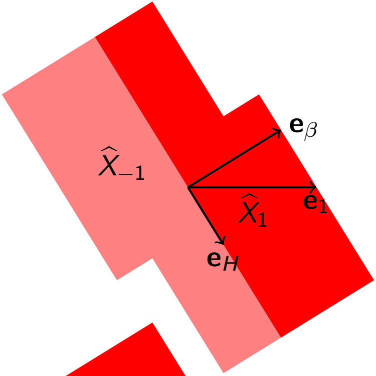
$\beta = \frac{1+\sqrt{5}}{2}$

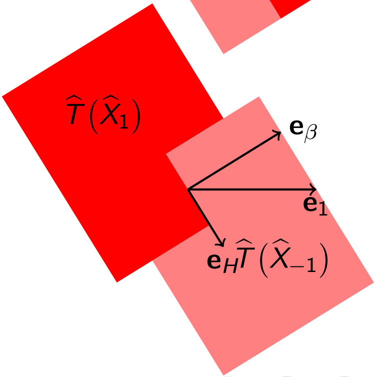
$A = \{-1, 1\}$

$X_{-1} = [-1, 0),\ X_1 = [0, 1)$

$\Rightarrow$ multiple tiling of degree 4







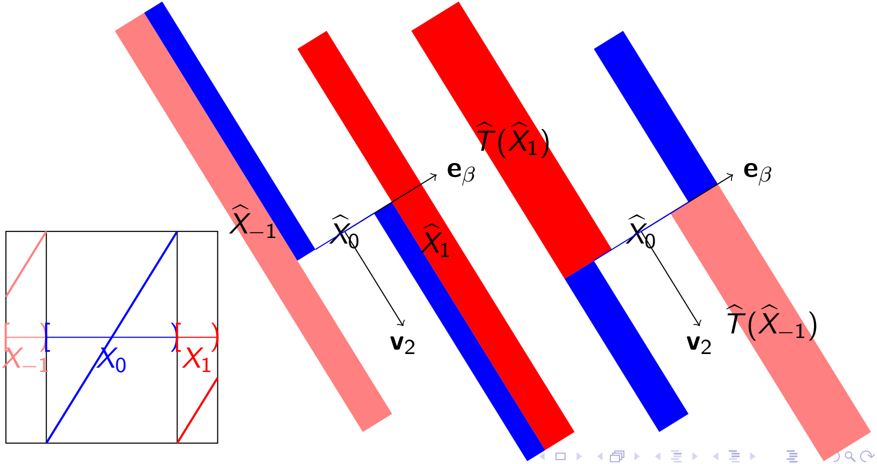
Symmetric β-transformations (Akiyama–Scheicher 2007)

$X = \big[-\frac{1}{2}, \frac{1}{2}\big)$, $T(x) = \beta x - \big\lfloor \beta x + \frac{1}{2} \big\rfloor$

$\beta \le 3$: $A = \{-1, 0, 1\}$,

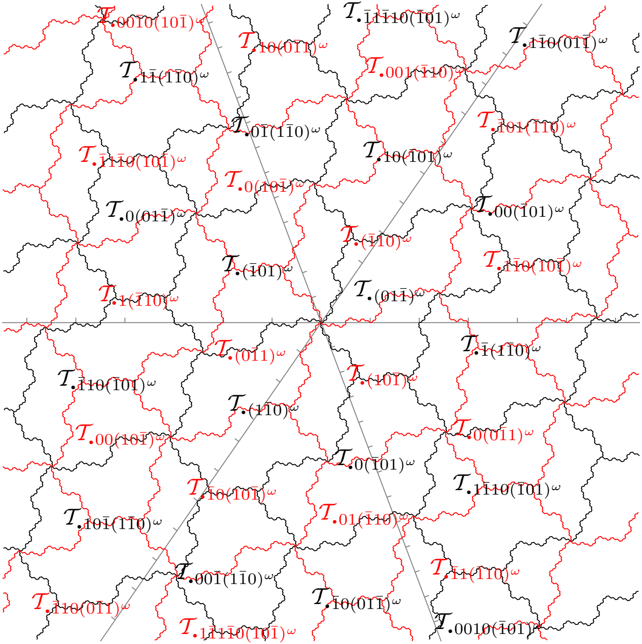
$X_{-1} = \big[-\frac{1}{2}, -\frac{1}{2\beta}\big)$, $X_0 = \big[-\frac{1}{2\beta}, \frac{1}{2\beta}\big)$, $X_1 = \big[\frac{1}{2\beta}, \frac{1}{2}\big)$

Example: $\beta = \frac{1+\sqrt{5}}{2} \Rightarrow$ tiling



symmetric β-transformation, $\beta^3 = \beta^2 + \beta + 1$

$\Rightarrow$ double tiling



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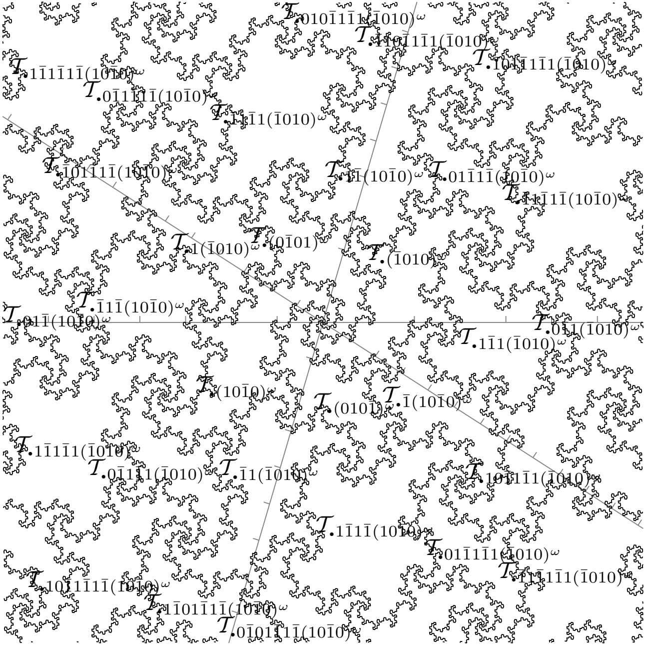
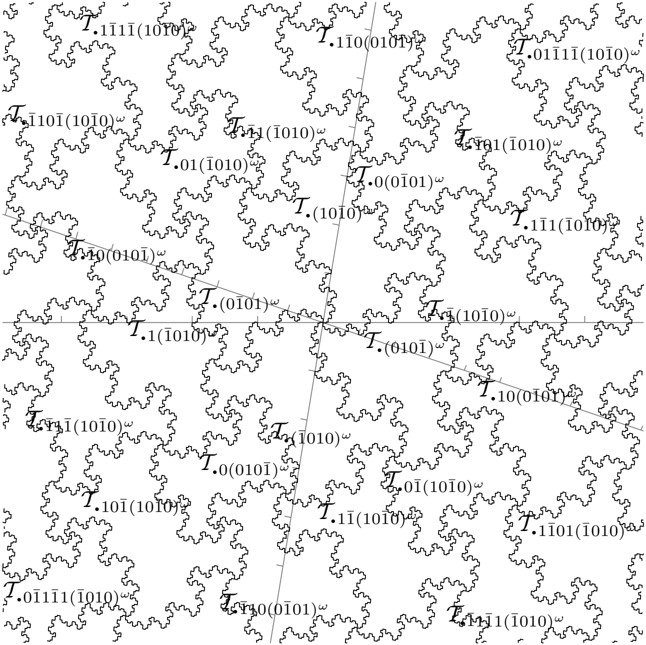


symmetric β-transformation, $\beta^3 = 2\beta^2 - \beta + 1$

$\Rightarrow$ tiling



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