Écritures de nombres en base réelle, fractals et pavages

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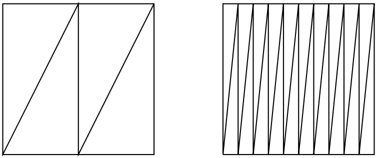
Digital expansions in base β

Let $\beta \geq 2$ be an integer. The β-expansion (binary, ternary, decimal, ...) of $x \in [0,1)$ is given by the β-transformation

$$T_\beta : [0,1) \to [0,1), \; x \mapsto T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$



$\beta = 2$ $\qquad$ $\beta = 10$

where $\lfloor y \rfloor = \max\{n \in \mathbb{Z} \mid n \leq y\}$.

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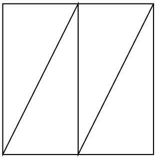

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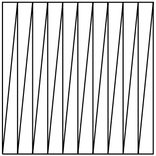

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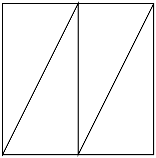

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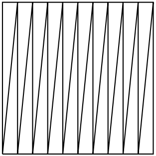

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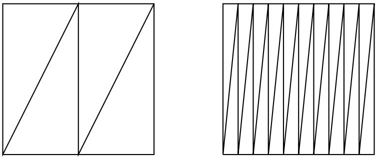

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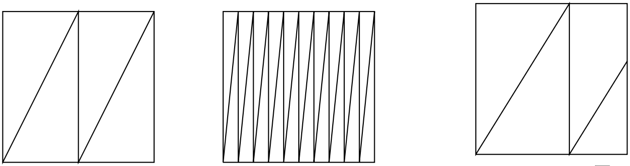
with $b_n = \lfloor \beta T_\beta^{n-1}(x) \rfloor \in \{0, 1, \ldots, \beta - 1\}$. Set $d_\beta(x) = b_1 b_2 \cdots$.

Digital expansions in base β

Let $\beta > 1$ be a real number. The (greedy) β-expansion of $x \in [0,1)$ is given by the β-transformation

$$T_\beta : [0,1) \to [0,1), \; x \mapsto T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$



$\beta = 2$ $\quad$ $\beta = 10$ $\quad$ $\beta = (1+\sqrt{5})/2$

where $\lfloor y \rfloor = \max\{n \in \mathbb{Z} \mid n \leq y\}$. We have

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Admissible sequences

The infinite expansion of 1 in base β is $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$, where $a_n = \lceil \beta \widetilde{T}_\beta^{n-1}(1) \rceil - 1$ is given by the transformation

$$\widetilde{T}_\beta : (0,1] \to (0,1], \; x \mapsto \widetilde{T}_\beta(x) = \beta x - \big(\lceil \beta x \rceil - 1\big)$$

and $\lceil y \rceil = \min\{n \in \mathbb{Z} \mid n \geq y\}$.

Theorem (Parry 1960)

We have $b_1 b_2 \cdots = d_\beta(x)$ for some $x \in [0,1)$ if and only if

$$b_n \in \mathbb{N} \quad \textit{and} \quad b_n b_{n+1} \cdots <_{\mathrm{lex}} a_1 a_2 \cdots \qquad \forall n \geq 1.$$

Such a sequence $b_1 b_2 \cdots$ is called β-admissible.

Examples

$\beta \in \mathbb{N}$: $a_1 a_2 \cdots = (\beta - 1)^\omega$, every sequence in $\{0, 1, \ldots, \beta - 1\}^\omega$ not terminating by $(\beta - 1)^\omega$ is β-admissible

$\beta = (1 + \sqrt{5})/2$: $a_1 a_2 \cdots = (1\,0)^\omega$, every sequence in $\{0, 1\}^\omega$ without $1\,1$ and not terminating by $(1\,0)^\omega$ is β-admissible

Periodic β-expansions for Pisot numbers β

Pisot number: algebraic integer $\beta > 1$ with $|\alpha| < 1$ for every Galois conjugate $\alpha \neq \beta$; in particular every integer $\beta \geq 2$

If $\beta \geq 2$ is an integer, then $d_\beta(x)$ is eventually periodic if and only if $x \in \mathbb{Q} \cap [0, 1)$, $d_\beta(x)$ is purely periodic if and only if the denominator of x is coprime with β.

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Theorem (Schmidt 1980)

If β is Pisot, $d_\beta(x)$ is eventually periodic iff $x \in \mathbb{Q}(\beta) \cap [0, 1)$.

If $d_\beta(x)$ is eventually periodic for every $x \in \mathbb{Q} \cap [0, 1)$, then β is Pisot or Salem ($|\alpha| \leq 1$ for every Galois conjugate $\alpha \neq \beta$).

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If $\beta^2 - n\beta - 1 = 0$ for some $n \in \mathbb{Z}$, $n \geq 1$, then $d_\beta(x)$ is purely periodic for every $x \in \mathbb{Q} \cap [0, 1)$.

Lemma (Akiyama 1998)

If β has a positive Galois conjugate (in particular if $\beta^2 - n\beta + 1 = 0$), then $d_\beta(x)$ is not purely periodic for any $x \in \mathbb{Q} \cap (0, 1)$.

Natural extension of T_β for Pisot units β

Let β be a Pisot number, M_β a companion matrix to the minimal polynomial $X^d - c_1X^{d-1} - c_2X^{d-2} - \cdots - c_d \in \mathbb{Z}[X]$ of β,

$$M_\beta = \begin{pmatrix} c_1 & c_2 & \cdots & \cdots & c_d \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

M_β is expanding by the factor β on $E_\beta = \mathbb{R}(\beta^{d-1}, \ldots, \beta, 1)^t$, contracting on a hyperplane H of $\mathbb{R}^d$ (spanned by the eigenvectors corresponding to the conjugates of β).
Let π be the projection on E_β along H and

$$\mathbf{e}_1 = (1, 0, \ldots, 0)^t = \mathbf{e}_\beta - \mathbf{e}_H$$

with $\mathbf{e}_\beta = \pi(\mathbf{e}_1) \in E_\beta$, $\mathbf{e}_H \in H$.

Natural extension of T_β for Pisot units β

Let $\mathbf{e}_1 = \mathbf{e}_\beta - \mathbf{e}_H$,

$$\mathcal{S}_\beta = \big\{(b_n)_{n\in\mathbb{Z}} \mid b_n b_{n+1}\cdots \text{ is } \beta\text{-admissible } \forall n \in \mathbb{Z}\big\},$$

$$\psi:\ \mathcal{S}_\beta \to \mathbb{R}^d,\ (b_n)_{n\in\mathbb{Z}} \mapsto \underbrace{\sum_{n=1}^{\infty} b_n\beta^{-n}}_{\in[0,1)}\mathbf{e}_\beta + \underbrace{\sum_{n=-\infty}^{0} b_n M_\beta^{-n}\mathbf{e}_H}_{\in H},$$

$$\widehat{T}_\beta:\ \mathcal{X}_\beta = \psi(\mathcal{S}_\beta) \to \mathcal{X}_\beta,\ \mathbf{x} \mapsto M_\beta\mathbf{x} - b_1\mathbf{e}_1.$$

For $\mathbf{x} = x\mathbf{e}_\beta + \mathbf{y}$, $x \in [0,1)$, $\mathbf{y} \in H$, we have

$$\widehat{T}_\beta(x\mathbf{e}_\beta + \mathbf{y}) = \underbrace{(\beta x - b_1)}_{T_\beta(x)}\mathbf{e}_\beta + M_\beta\mathbf{y} + b_1\mathbf{e}_H,$$

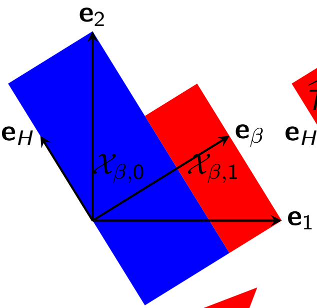
thus $\widehat{T}_\beta \circ \psi = \psi \circ \sigma$, where σ is the left-shift, and $\pi \circ \widehat{T}_\beta = T_\beta \circ \pi$.

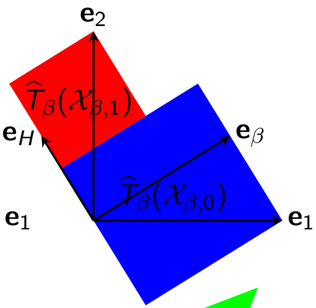
If β is a Pisot unit ($|\det M_\beta| = |c_d| = 1$), then $\widehat{T}_\beta$ is bijective except on a set of measure 0, $(\widehat{T}_\beta, \mathcal{X}_\beta)$ is a natural extension of $(T_\beta, [0,1))$.

$\widehat{T}_\beta$ is a toral automorphism since $\widehat{T}_\beta(\mathbf{x}) \equiv M_\beta\mathbf{x} \pmod{\mathbb{Z}^d}$.

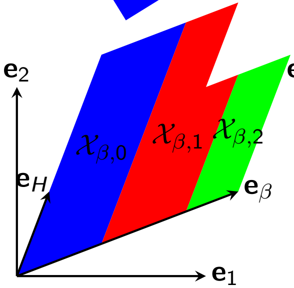
Natural extensions for quadratic Pisot units β

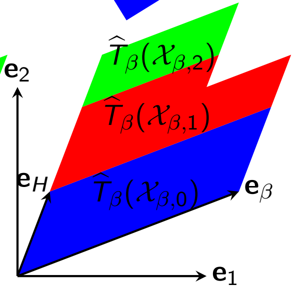
$\beta^2 = \beta + 1$
$\beta \approx 1.618$
(golden mean)





$\beta^2 = 3\beta - 1$
$\beta \approx 2.618$
(square of the golden mean)

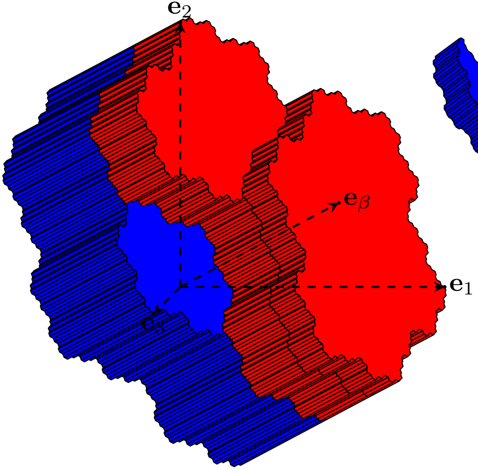




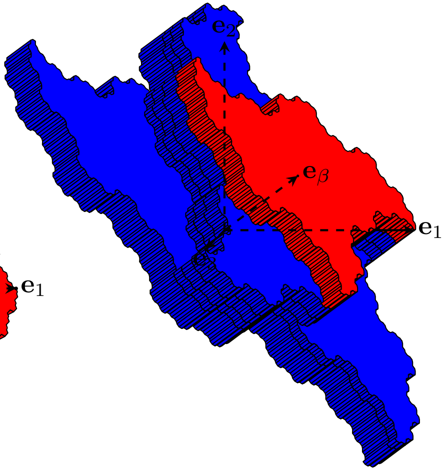
$$\mathcal{X}_{\beta,k} = \psi\big(\{(b_n)_{n\in\mathbb{Z}} \in \mathcal{S}_\beta \mid b_1 = k\}\big),\ \widehat{T}(\mathcal{X}_{\beta,k}) = M_\beta \mathcal{X}_{\beta,k} - k\mathbf{e}_1$$

Natural extensions for cubic Pisot units β

$\beta^3 = \beta^2 + \beta + 1$, $\beta \approx 1.8393$
(Tribonacci number)



$\beta^3 = \beta + 1$, $\beta \approx 1.3247$
(smallest Pisot number)



Shape of $\mathcal{X}_\beta$

Since $\mathcal{X}_\beta = \psi(\mathcal{S}_\beta)$ with

$$\mathcal{S}_\beta = \big\{(b_n)_{n\in\mathbb{Z}} \mid b_n b_{n+1}\cdots \text{ is } \beta\text{-admissible } \forall n\in\mathbb{Z}\big\},$$

$$\psi:\ \mathcal{S}_\beta \to \mathbb{R}^d,\ (b_n)_{n\in\mathbb{Z}} \mapsto \sum_{n=1}^{\infty} b_n \beta^{-n}\mathbf{e}_\beta + \sum_{n=-\infty}^{0} b_n M_\beta^{-n}\mathbf{e}_H,$$

we have

$$\mathcal{X}_\beta = \bigcup_{x\in[0,1)} \big(x\mathbf{e}_\beta + \mathcal{D}_\beta(x)\big),$$

where

$$\mathcal{D}_\beta(x) = \left\{ \sum_{n=-\infty}^{0} b_n M_\beta^{-n}\mathbf{e}_H \,\middle|\, (b_n)_{n\in\mathbb{Z}} \in \mathcal{S}_\beta,\ b_1 b_2\cdots = d_\beta(x) \right\}.$$

Lemma

If β is a Pisot number, then $\mathcal{V}_\beta = \big\{\widetilde{T}_\beta^n(1) \mid n\geq 0\big\}$ is a finite set. We have $\mathcal{D}_\beta(x) \supseteq \mathcal{D}_\beta(y)$ if $0\leq x\leq y<1$, with $\mathcal{D}_\beta(x)=\mathcal{D}_\beta(y)$ if and only if $[x,y)\cap\mathcal{V}_\beta=\emptyset$, hence $\#\{\mathcal{D}_\beta(x)\mid x\in[0,1)\} = \#\mathcal{V}_\beta$. $\mathcal{D}_\beta(x)$ is compact for every $x\in[0,1)$.

Determining digits in $d_\beta(x)$

Let β be a Pisot unit, $x \in [0,1)$ and $d_\beta(x) = b_1 b_2 \cdots$.

We have $b_n = k$ if and only if $\widehat{T}_\beta^{n-1}(x\mathbf{e}_\beta) \in \mathcal{X}_{\beta,k}$. Note that

$$x\beta^{n-1}\mathbf{e}_\beta = M_\beta^{n-1}(x\mathbf{e}_\beta) \equiv \widehat{T}_\beta^{n-1}(x\mathbf{e}_\beta) \pmod{\mathbb{Z}^d},$$

thus $x\beta^{n-1}\mathbf{e}_\beta \in \mathcal{X}_{\beta,k} \pmod{\mathbb{Z}^d}$ if $b_n = k$.

Conjecture

If β is a Pisot unit, then the intersection of $\mathcal{X}_{\beta,k}$ and $\mathcal{X}_{\beta,\ell}$ mod $\mathbb{Z}^d$ *has Lebesgue measure zero for every $\ell \neq k$.*

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A (Pisot) number $\beta > 1$ is said to satisfy (F) if $d_\beta(x)$ is finite (terminates with 0^ω) for every $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$.

Theorem

If β is a Pisot unit satisfying (F), then, for every $x \in [0,1)$, $n \geq 1$, $b_n = k$ if and only if $x\beta^{n-1}\mathbf{e}_\beta \in \mathcal{X}_{\beta,k} \pmod{\mathbb{Z}^d}$.

It is easy to determine for any β if (F) holds, some classes of numbers satisfying (F) are known (Frougny–Solomyak 1992: $c_1 \geq c_2 \geq \cdots \geq c_d > 0$, Hollander 1996), only quadratic numbers (Frougny–Solomyak 1992) and cubic units (Akiyama 2000) are completely classified.

Tilings for Pisot units β

Let β be a Pisot unit. For every $x \in \mathbb{Z}[\beta] \cap [0,1)$, let

$$\mathcal{T}_\beta(x) = \Phi(x) + \mathcal{D}_\beta(x) \quad \text{with} \quad \Phi(x) = x\mathbf{e}_\beta - \Xi(x) \in H.$$

We have $\mathcal{T}_\beta(x) \equiv x\mathbf{e}_\beta + \mathcal{D}_\beta(x) \pmod{\mathbb{Z}^d}$ since $\Xi(x) \in \mathbb{Z}^d$.
Remember that $\mathcal{X}_\beta = \bigcup_{x\in[0,1)} \big(x\mathbf{e}_\beta + \mathcal{D}_\beta(x)\big)$.

Theorem (Ito–Rao 2006, Ei–Ito–Rao 2006, Berthé–Siegel 2005)

The family $\{\mathcal{T}_\beta(x)\}_{x\in\mathbb{Z}[\beta]\cap[0,1)}$ forms a quasi-periodic multiple tiling of H. It is a tiling if and only if $\{\overline{\mathcal{X}_\beta} + \mathbf{z}\}_{\mathbf{z}\in\mathbb{Z}^d}$ forms a tiling of $\mathbb{R}^d$. If these are tilings and $\#\mathcal{V}_\beta = d$, then $\mathcal{T}_\beta(0)$ tiles H periodically.

(Conjecture: $\{\mathcal{T}_\beta(x)\}_{x\in\mathbb{Z}[\beta]\cap[0,1)}$ forms a tiling for every Pisot unit β)

Multiple tiling:

- finitely many sets up to translation,
- every set is compact and the closure of its interior,
- $\exists m \geq 1$ such that almost every point lies in exactly m sets.

Tiling: $m = 1$

Tilings for Pisot units β

Lemma

$\{\mathcal{T}_\beta(x)\}_{x\in\mathbb{Z}[\beta]\cap[0,1)}$ *forms a tiling of H if and only if there exists an exclusive point* $\mathbf{y} \in \mathcal{T}_\beta(0)$, *i.e.,* $\mathbf{y} \notin \mathcal{T}_\beta(x)$ *for every* $x \in \mathbb{Z}[\beta] \cap (0,1)$.

Let $P_\beta = \big\{x \in \mathbb{Z}[\beta] \cap [0,1) \mid d_\beta(x) \text{ is purely periodic}\big\}$.

Note that $\Xi(P_\beta) = \mathbb{Z}^d \cap \mathcal{X}_\beta$.

Lemma

$\mathbf{0} \in \mathcal{T}_\beta(x)$ *if and only if* $x \in P_\beta$.

$\mathbf{0}$ *is an exclusive point of* $\mathcal{T}_\beta(0)$ *if and only if* $P_\beta = \{0\}$, *i.e., (F) holds.*

Theorem (Akiyama 2002, Kalle–St)

$\mathcal{T}_\beta(0)$ *has an exclusive point if and only if there exists* $y \in \mathbb{Z}[\beta] \cap [0,\varepsilon)$, $\varepsilon = \min_{x\in P_\beta}\big(1 + \lfloor \beta x \rfloor - \beta x\big)$, *such that* $d_\beta(x+y)$ *is finite* $\forall x \in P_\beta$.

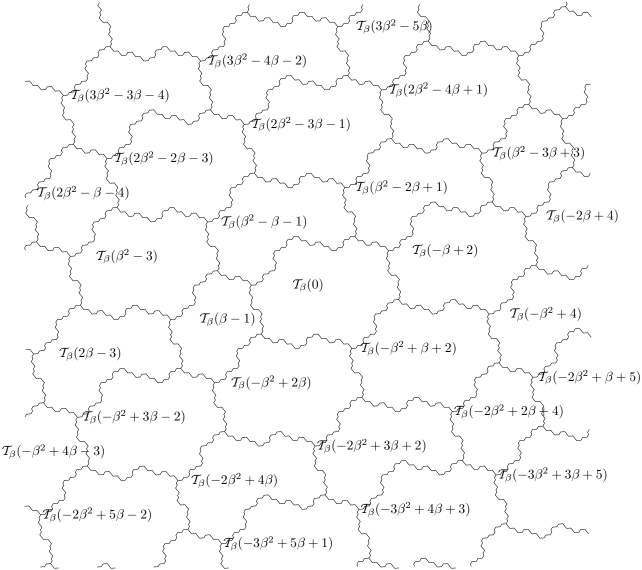
(cf. (W) property, conjectured to be true for every Pisot number β)

Theorem (Akiyama–Rao–St 2004)

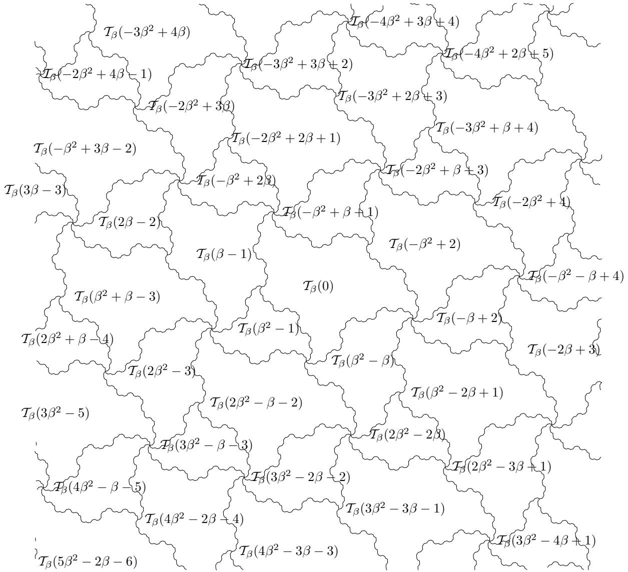
If $c_1 > |c_2| + \cdots + |c_d|$, *then* $\{\mathcal{T}_\beta(x)\}_{x\in\mathbb{Z}[\beta]\cap[0,1)}$ *forms a tiling of H.*

(Barge–Kwapisz 2006: tiling property for another class of β)

Tiling of H for the Tribonacci number ($\beta^3 = \beta^2 + \beta + 1$)



Tiling of H for the smallest Pisot number ($\beta^3 = \beta + 1$)



Purely periodic β-expansions for Pisot units β

For $x \in \mathbb{Q}(\beta)$, let $\Xi(x) \in \mathbb{Q}^d$ be the sum of all conjugates of $x\mathbf{e}_\beta$, in particular $\Xi(r) = r\mathbf{e}_1$ for $r \in \mathbb{Q}$.

Theorem (Ito–Rao 2005)

If β is a Pisot unit, then $d_\beta(x)$ is purely periodic if and only if $x \in \mathbb{Q}(\beta) \cap [0,1)$ and $\Xi(x) \in \mathcal{X}_\beta$. In particular, $d_\beta(r)$ is purely periodic for $r \in \mathbb{Q} \cap [0,1)$ if and only if $r\mathbf{e}_1 \in \mathcal{X}_\beta$.

(Berthé–Siegel 2007: characterization for Pisot non-units β)

Corollary

Let β be a Pisot unit and

$$\gamma_\beta = \sup\left\{x \in [0,1) \mid d_\beta(r) \text{ is purely periodic } \forall r \in [0,x] \cap \mathbb{Q}\right\}.$$

Then $\gamma_\beta \mathbf{e}_1$ is on the boundary of $\mathcal{X}_\beta$, hence $\gamma_\beta \in \left\{\widetilde{T}_\beta^n(1) \mid n \geq 0\right\}$ or $-\gamma_\beta \mathbf{e}_H$ is on the boundary of $\mathcal{D}_\beta(\gamma_\beta)$.

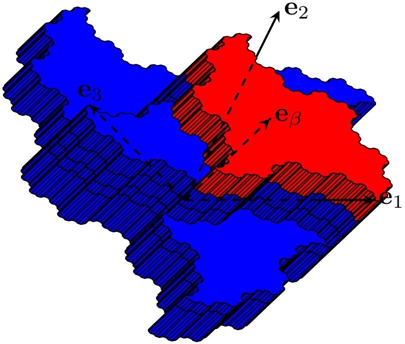
(Akiyama–Barat–Berthé–Siegel 2008: γ_β for Pisot non-units β)

β has a positive conjugate, in particular $\beta^2 = n\beta - 1 \Rightarrow \gamma_\beta = 0$, $\beta^2 = n\beta + 1 \Rightarrow \gamma_\beta = 1$.

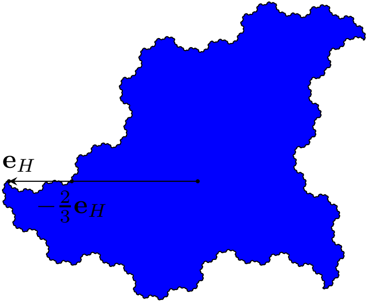
γ_β for the smallest Pisot number ($\beta^3 = \beta + 1$)

Akiyama–Scheicher 2005: $\gamma_\beta = 0.666666666086440674 88\cdots$

$\mathcal{X}_\beta$



$\mathcal{D}_\beta(x)$ for $x \in [\beta^{-2}, \beta^{-1}) \supset [0.57, 0.75]$



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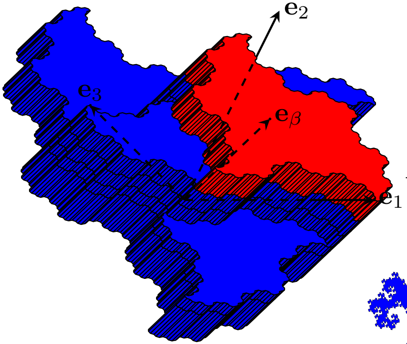
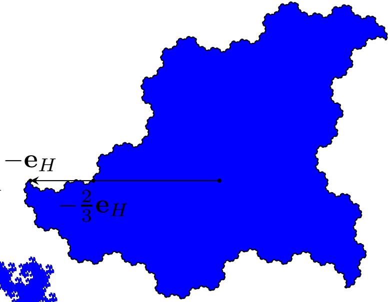
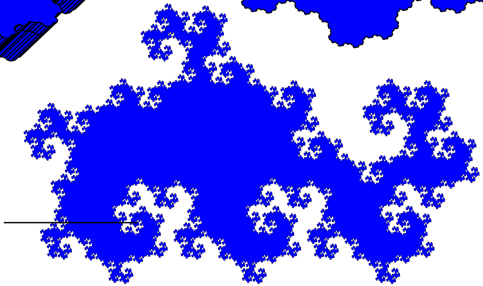
$\mathcal{X}_\beta$ $\qquad\qquad$ $\mathcal{D}_\beta(x)$ for $x \in [\beta^{-2}, \beta^{-1}) \supset [0.57, 0.75]$





$\mathcal{D}_\beta(0)$ for
$\beta^3 = 3\beta^2 - 2\beta + 1$:

$\Rightarrow \gamma_\beta = 0$



γ_β and (F)

Theorem (Akiyama 1998, 1999)

If β is a Pisot unit satisfying (F), then $\gamma_\beta > 0$.

Theorem (Adamczewski–Frougny–Siegel–St)

If β is a cubic Pisot unit, then $\gamma_\beta > 0$ is equivalent with (F).

Theorem (Adamczewski–Frougny–Siegel–St)

If β is a cubic Pisot unit satisfying (F) and β has a Galois conjugate $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then $\gamma_\beta \notin \mathbb{Q}$.

Idea of the proof: If $\gamma_\beta \in \left\{ \widetilde{T}_\beta^n(1) \mid n \geq 0 \right\} \setminus \{1\}$, then $\gamma_\beta \notin \mathbb{Q}$. Otherwise, $-\gamma_\beta \mathbf{e}_H$ is on the boundary of $\mathcal{D}_\beta(\gamma_\beta)$. Every $\mathcal{D}_\beta(x)$ is the solution of a graph-directed Iterated Function System, consisting of contracting similitudes with irrational rotation. Using the self-similarity, one can show that every point $-r\mathbf{e}_H$, $r \in \mathbb{Q}$, on the boundary of $\mathcal{D}_\beta(x)$ is a "spiral point", which means that there are both intervals in $\mathcal{D}_\beta(x)$ and in its complement arbitarily close to $-r\mathbf{e}_H$ in every direction. Thus $\gamma_\beta \neq r$. □

β-expansions of minimal weight

A word $b_1 \cdots b_n \in \mathbb{Z}^*$ is a β-expansion of minimal weight if $\sum_{j=1}^{n} |b_j| \leq \sum_{j=1}^{m} |c_j|$ for any word $c_1 \cdots c_m \in \mathbb{Z}^*$ satisfying

$$\sum_{j=1}^{n} \frac{b_j}{\beta^j} = \beta^k \sum_{j=1}^{m} \frac{c_j}{\beta^j} \text{ for some } k \in \mathbb{Z}.$$

(Frougny–St 2008: If β is a Pisot number, then the set of β-expansions of minimal weight is recognizable by a finite automaton.)

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For $1 < \beta \le 3$ and $\frac{1}{2} \le \alpha \le \frac{1}{\beta-1}$, let

$$T_{\beta,\alpha}: [-\alpha,\alpha) \to [-\alpha,\alpha),\ x \mapsto \beta x - \begin{cases} 1 & \text{if } x \in [\alpha/\beta, \alpha), \\ 0 & \text{if } x \in [-\alpha/\beta, \alpha/\beta), \\ -1 & \text{if } x \in [-\alpha, -\alpha/\beta). \end{cases}$$

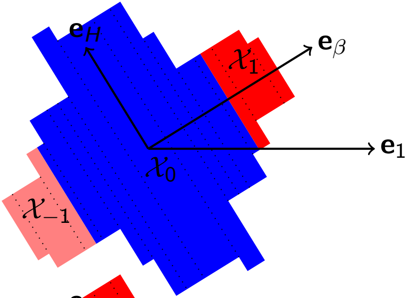
and define $d_{\beta,\alpha}(x)$ for $x \in [-\alpha,\alpha)$ similarly to $d_\beta(x)$.

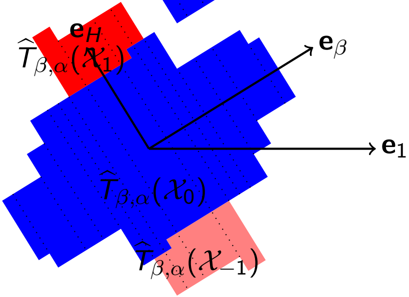
Theorem (Frougny–St 2008)

If $\beta^2 = \beta + 1$ and $\frac{\beta^2}{\beta^2+1} \le \alpha \le \frac{2\beta}{\beta^2+1}$, or $\beta^3 = \beta^2 + \beta + 1$ and $\frac{\beta}{\beta+1} \le \alpha \le \frac{2+1/\beta}{\beta+1}$, or $\beta^3 = \beta + 1$ and $\frac{\beta^3}{\beta^2+1} \le \alpha \le \frac{\beta^2+1/\beta}{\beta^2+1}$, then every prefix of $d_{\beta,\alpha}(x)$ for any $x \in [-\alpha,\alpha)$ is a β-expansion of minimal weight.

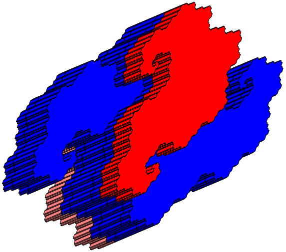
Natural extensions for β-expansions of minimal weight

$\beta = (1+\sqrt{5})/2$, $\alpha = \frac{\beta^2+\beta^{-3}}{\beta^2+1}$

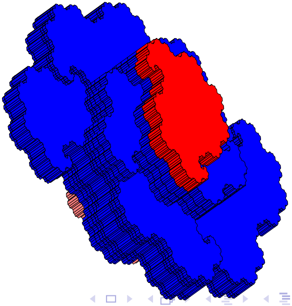




$\beta^3 = \beta^2+\beta+1$, $\alpha = \frac{\beta}{\beta+1}$



$\beta^3 = \beta+1$, $\alpha = \frac{\beta^3}{\beta^2+1}$



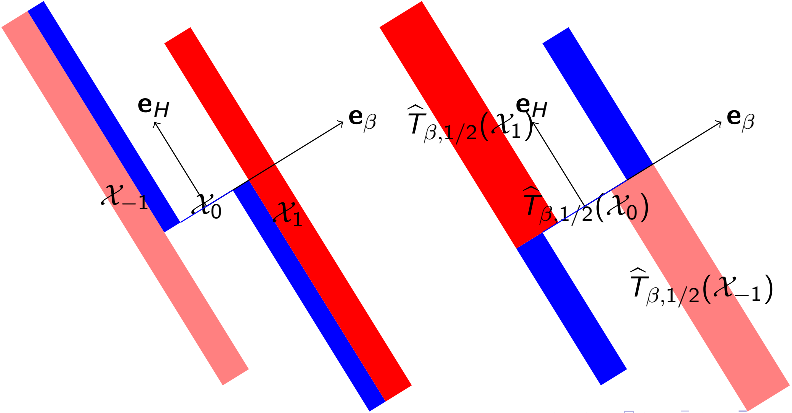
Symmetric β-transformations (Akiyama–Scheicher 2007)

For $\beta > 1$, the symmetric β-transformation is defined by

$$S_\beta : [-1/2, 1/2) \to [-1/2, 1/2), \; x \mapsto \beta x - \lfloor \beta x + 1/2 \rfloor.$$

(For $\beta \leq 3$, we have $S_\beta = T_{\beta,\alpha}$.)

Example: $\beta = (1+\sqrt{5})/2$, natural extension domain $\Rightarrow$ tiling



Double tiling of H by the symmetric β-transformation for the Tribonacci number ($\beta^3 = \beta^2 + \beta + 1$)