

Discrepancy of substitutive van der Corput sequences

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van der Corput sequences

digital expansion of $n \in \mathbb{N}$ in base $b \in \mathbb{N}$, $b \geq 2$:

$$n = \sum_{j=1}^k \epsilon_j b^{j-1} = \epsilon_k \cdots \epsilon_2 \epsilon_1, \quad \epsilon_j \in \{0, 1, \dots, b-1\}$$

van der Corput-sequence in base b :

$$x_n = \sum_{j=1}^k \epsilon_j b^{-j} = .\epsilon_1 \epsilon_2 \cdots \epsilon_k$$

n	1	2	3	4	5	6	7	...
$b = 2$: bin. exp.	1	10	11	100	101	110	111	...
x_n	.1	.01	.11	.001	.101	.011	.111	...

β -adic van der Corput sequences

greedy expansion of $n \in \mathbb{N}$ with Fibonacci numbers:

$$(F_0 = 1, F_1 = 2, F_j = F_{j-1} + F_{j-2})$$

$$n = \sum_{j=1}^k \epsilon_j F_{j-1} = \epsilon_k \cdots \epsilon_2 \epsilon_1 \text{ with } \sum_{j=1}^{\ell} \epsilon_j F_{j-1} < F_{\ell}, \epsilon_j \in \{0, 1\}$$
$$(\epsilon_j = 1 \Rightarrow \epsilon_{j-1} = 0)$$

β -adic van der Corput sequence, $\beta = \frac{1+\sqrt{5}}{2}$:

$$x_n = \sum_{j=1}^k \epsilon_j \beta^{-j} = .\epsilon_1 \epsilon_2 \cdots \epsilon_k$$

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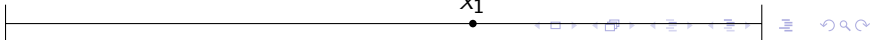
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x_2

x_1

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x_3

x_2

x_1

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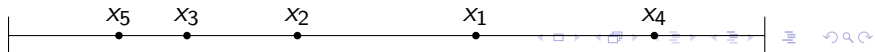
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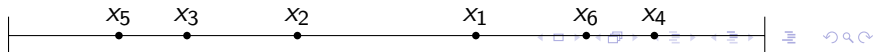
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x_5

x_3

x_2

x_7

x_1

x_6

x_4

β -adic van der Corput sequences

greedy β -expansion of $x \in [0, 1)$, $\beta > 1$:

$$x = \sum_{j=1}^{\infty} \epsilon_j \beta^{-j} = .\epsilon_1 \epsilon_2 \dots \text{ with } \sum_{j=\ell+1}^{\infty} \epsilon_j \beta^{-j} < \beta^{-\ell}, \epsilon_j \in \{0, 1, \dots, \lfloor \beta \rfloor\}$$

$$S_\beta = \{\epsilon_k \dots \epsilon_2 \epsilon_1 \mid .\epsilon_1 \epsilon_2 \dots \epsilon_k \text{ is a greedy expansion, } \epsilon_k > 0\}$$

shortlex order on S_β : $\epsilon_k \dots \epsilon_2 \epsilon_1 < \epsilon'_{k'} \dots \epsilon'_2 \epsilon'_1$ if

$$k < k' \text{ or } k = k', \epsilon_k \dots \epsilon_{\ell+1} = \epsilon'_k \dots \epsilon'_{\ell+1}, \epsilon_\ell < \epsilon'_\ell$$

β -adic van der Corput sequence, $\beta > 1$ (Ninomiya 1998):

$$x_n = .\epsilon_1 \epsilon_2 \dots \epsilon_k, \text{ where } \epsilon_k \dots \epsilon_2 \epsilon_1 \text{ is the } n\text{-th element of } S_\beta$$

β -adic van der Corput sequences

Ninomiya (1998):

If β is a Pisot number with irreducible β -polynomial, then the β -adic van der Corput sequence is a low discrepancy sequence.

Can we calculate x_n directly from a digital expansion of n ?

$\beta = 1 + \sqrt{2}$ ($\beta^2 = 2\beta + 1$), $G_0 = 1, G_1 = 3, G_j = 2G_{j-1} + G_{j-2}$:

n	1	2	3	4	5	6	...
G -exp.	1	2	10	11	12	20	...
x_n	.1	.2	.01	.11	.02	.12	...

(language of greedy expansions not closed under reversal)

Dumont-Thomas expansions of real numbers

σ substitution (morphism) on $\{1, 2, \dots, d\}^*$, $\sigma(1)$ starts with 1
 $M_\sigma = (|\sigma(j)|_i)_{1 \leq i, j \leq d}$ incidence matrix of σ with dominant eigenvalue $\beta > 1$, left eigenvector $(\eta(1), \dots, \eta(d))$, $\eta(1) = 1$
($|w|_i$ denotes the number of letters i in a word w)

Example:

$$\begin{array}{lcl} \sigma : 1 & \mapsto & 1223 \\ & 2 & \mapsto 122 \\ & 3 & \mapsto 222 \end{array} \quad M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix} \quad \beta^3 = 3\beta^2 + 3$$

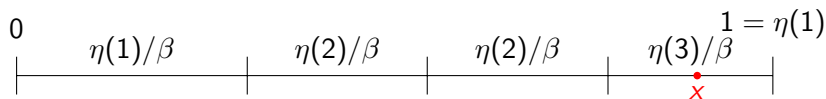
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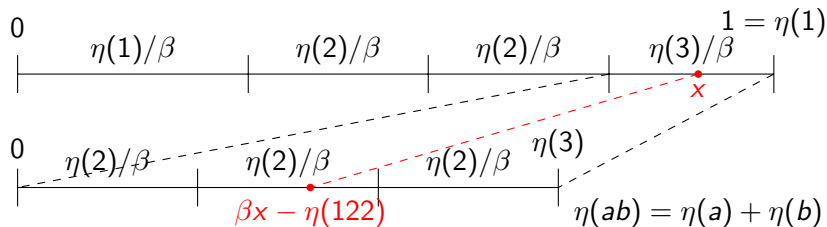
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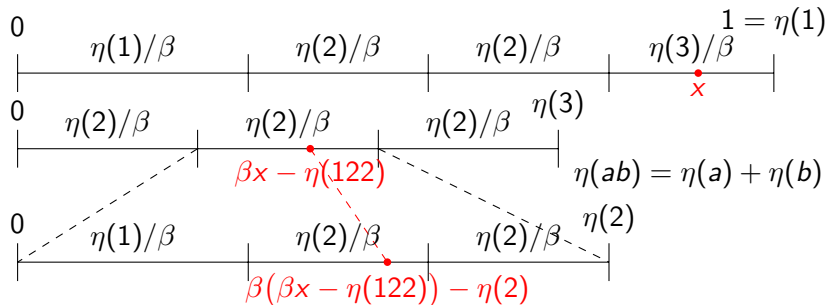


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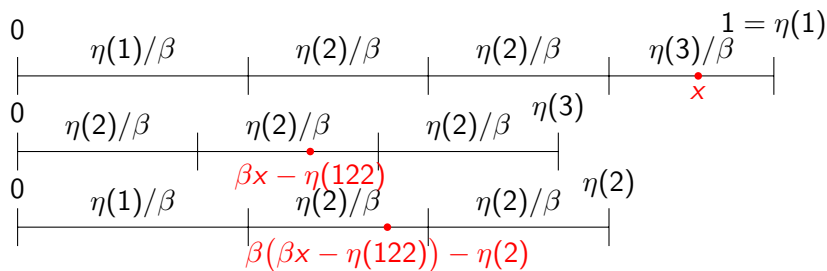
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Dumont-Thomas expansions of real numbers



$$x = \frac{\eta(122)}{\beta} + \frac{\eta(2)}{\beta^2} + \frac{\eta(1)}{\beta^3} + \dots = .\eta(122)\eta(2)\eta(1)\dots$$

A sequence of words m_1, m_2, \dots is σ -admissible if there exists a companion sequence of letters $b_0 = 1, b_1, b_2, \dots$ such that

$$m_j b_j \text{ is a prefix of } \sigma(b_{j-1}) \text{ for all } j \geq 1.$$

Every $x \in [0, 1)$ has an expansion $x = .\eta(m_1)\eta(m_2)\dots$ with some σ -admissible sequence m_1, m_2, \dots

Substitutive van der Corput sequences

Order the sequences $|m_k| \cdots |m_2| |m_1|$ such that m_1, m_2, \dots, m_k is σ -admissible (and $|m_k| > 0$) by the shortlex order.

($|w|$ denotes the length of w)

Then the substitutive van der Corput sequence is defined by

$$x_n = \cdot \eta(m_1) \eta(m_2) \cdots \eta(m_k)$$

if $|m_k| \cdots |m_2| |m_1|$ is the n -th such sequence.

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Special case of β -substitutions:

$$\begin{aligned} \sigma : 1 &\mapsto 1^{t_1}2 \\ 2 &\mapsto 1^{t_2}3 \quad \text{with } t_{\ell+1} \cdots t_d (t_{d-p+1} \cdots t_d)^\infty \\ &\vdots \\ &< t_1 \cdots t_d (t_{d-p+1} \cdots t_d)^\infty \text{ for } 1 \leq \ell < d \\ d-1 &\mapsto 1^{t_{d-1}}d \\ d &\mapsto 1^{t_d}(d-p+1), \quad 1 \leq p \leq d \end{aligned}$$

Then $1 = .t_1 \cdots t_{d-p}(t_{d-p+1} \cdots t_d)^\infty$, i.e.,

$$\beta^d - t_1\beta^{d-1} - \dots - t_d = \beta^{d-p} - t_1\beta^{d-p+1} - \dots - t_{d-p}.$$

All words m_j are of the form 1^{ϵ_j} , thus $\eta(m_j) = |m_j| = \epsilon_j$ and $(x_n)_{n \geq 1}$ is the β -adic van der Corput sequence.

Dumont-Thomas expansions of integers

σ substitution on $\{1, 2, \dots, d\}$, $\sigma(1)$ starts with 1
 $u = \sigma^\infty(1)$ fixed point of σ , $u[j]$ denotes the j -th letter of u

For every $n \in \mathbb{N}$, there exists a unique σ -admissible sequence m_1, m_2, \dots, m_k with $|m_1| > 0$ such that

$$u[1]u[2] \cdots u[n] = \sigma^{k-1}(m_1)\sigma^2(m_2) \cdots \sigma^0(m_k).$$

$$u = \sigma^k(1) \cdots = \sigma^{k-1}(m_1 b_1) \cdots = \sigma^{k-1}(m_1)\sigma^{k-2}(m_2 b_2) \cdots$$

Then $n = \sum_{j=1}^k |\sigma^{k-j}(m_j)|$ and we have $n < n'$ if and only if

$$|m_1| |m_2| \cdots |m_k| < |m'_1| |m'_2| \cdots |m'_{k'}| \quad (\text{shortlex order}).$$

Reverse substitution

σ is **ascending** if $|\sigma(b)| \geq |\sigma(c)|$ and $\sigma(b)[i] \leq \sigma(c)[j]$ for $b \leq c, i \leq j$.
If σ is ascending and primitive, then the **reverse substitution** τ is defined by

$$\tau(b)[j] = c \text{ if } \sigma(d+1-c)[j] \leq d+1-b \text{ and} \\ \sigma(d+2-c)[j] > d+1-b \text{ or } |\sigma(d+2-c)| < j \text{ or } c = 1.$$

(In the column j , the c -th line counted from below is the lowest line with a letter $\leq d+1-b$.)

Example:

$$\begin{array}{lcl} \sigma : 1 & \mapsto & 1223 \\ & 2 & \mapsto 122 \\ & 3 & \mapsto 222 \end{array} \quad \begin{array}{lcl} \tau : 1 & \mapsto & 1113 \\ & 2 & \mapsto 111 \\ & 3 & \mapsto 2 \end{array}$$

Proposition

The sequences $|m_1| |m_2| \cdots |m_k|$ such that m_1, m_2, \dots, m_k is σ -admissible are exactly the sequences $|m'_k| |m'_{k-1}| \cdots |m'_1|$ such that m'_1, m'_2, \dots, m'_k is τ -admissible.

Substitutive van der Corput sequences with ascending σ

If σ is ascending and primitive, then the substitutive van der Corput sequence is given by

$$x_n = \cdot \eta(m_1) \cdots \eta(m_k)$$

if the τ -Dumont-Thomas expansion of n is

$$n = \sum_{j=1}^k |\tau^{k-j}(m'_j)|$$

and m_1, \dots, m_k is the σ -admissible sequence such that

$$|m_1| \cdots |m_k| = |m'_k| \cdots |m'_1|.$$

Remark: After a permutation of letters, β -substitutions are ascending.

Discrepancy function

$$D(N, [0, y)) = A(N, [0, y)) - Ny$$

with $A(N, [0, y)) = \#\{x_n \in [0, y) : 0 \leq n < N\}$.

$$\begin{aligned} & D(N, [0, \cdot\eta(y_1)\eta(y_2)\cdots)) \\ &= \sum_{i=1}^{\infty} \left(A(N, [\cdot\eta(y_1)\cdots\eta(y_{i-1}), \cdot\eta(y_1)\cdots\eta(y_{i-1})\eta(y_i))) - N\eta(y_i)\beta^{-i} \right). \end{aligned}$$

If $N = \sum_{j=1}^k |\tau^{k-j}(m_j)|$, then

$$\begin{aligned} & A(N, [\cdot\eta(y_1)\cdots\eta(y_{i-1}), \cdot\eta(y_1)\cdots\eta(y_i))) \\ &= \mu_i(N, y) + \sum_{j=1}^{k-i} \sum_{\ell=1}^{|y_i|} \sum_{b=1}^{d+1-y_i[\ell]} |\tau^{k-i-j}(m_j)|_b \end{aligned}$$

with $0 \leq \mu_i(N, y) \leq \min(|m_i|, |y_i|) + 1$.

Substitutions of Pisot type

$$\begin{aligned} & A(N, [\cdot\eta(y_1) \cdots \eta(y_{i-1}), \cdot\eta(y_1) \cdots \eta(y_i)]) - N\eta(y_i)\beta^{-i} \\ &= \mu_i(N, y) + \sum_{j=1}^{k-i} \sum_{\ell=1}^{|y_i|} \sum_{b=1}^{d+1-y_i[\ell]} |\tau^{k-i-j}(m_j)|_b - \sum_{j=1}^k |\tau^{k-j}(m_j)|\eta(y_i)\beta^{-i} \end{aligned}$$

with $0 \leq \mu_i(N, y) \leq \min(|m_i|, |y_i|) + 1$, $N = \sum_{j=1}^k |\tau^{k-j}(m_j)|$.

If σ is of Pisot type (all eigenvalues $\lambda \neq \beta$ satisfy $|\lambda| \leq \rho < 1$), then

$$\begin{aligned} & \sum_{b=1}^{d+1-c} |\tau^{k-i-j}(m_j)|_b - \eta(c) |\tau^{k-j}(m_j)|\beta^{-i} = \mathcal{O}(\rho^{k-j}), \text{ hence} \\ & D\left(\sum_{j=1}^k |\tau^{k-j}(m_j)|, [0, \cdot\eta(y_1)\eta(y_2)\cdots]\right) = \mathcal{O}\left(\sum_{j=1}^k |m_j|\right) = \mathcal{O}(\log N), \\ & D\left(\sum_{j=1}^k |\tau^{k-j}(m_j)|, [0, \cdot\eta(y_1)\eta(y_2)\cdots]\right) = \mathcal{O}\left(\sum_{j=1}^{\infty} |y_j|\right). \end{aligned}$$

Substitutions of Pisot type

$$D\left(\sum_{j=1}^k |\tau^{k-j}(m_j)|, [0, \cdot\eta(y_1)\eta(y_2)\cdots]\right) = \mathcal{O}\left(\sum_{j=1}^k |m_j|\right) = \mathcal{O}(\log N)$$

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Theorem

If σ is an ascending substitution of Pisot type, then the substitutive van der Corput sequence is a low discrepancy sequence.

The interval $[0, y)$ is a bounded remainder set if and only if the σ -expansion of y is finite or it coincides eventually with the sequence m_1, m_2, \dots satisfying $m_j b_j = \sigma(b_{j-1})$.