Tree Automata Make Ordinal Theory Easy

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Plan:

Motivations, background

First Order theory of $(\omega^\omega, +)$, extension to $(\omega^{\omega^i}, +)$

Monadic Second Order theory of $(\omega^2, <)$, extension to $(\omega^i, <)$

Comparisons

Abstract

We give a new simple proof of the decidability of the First Order Theory of $(\omega^{\omega^i}, +)$ and the Monadic Second Order Theory of $(\omega^i, <)$, improving the complexity in both cases. Our algorithm is based on tree automata and a new representation of (sets of) ordinals by (infinite) trees.

Motivations

Automata based decision procedure are efficient and simple.

e.g. FO, MSO / $\mathbb{N}$, binary trees, . . .

Ordinal theories: original proof uses ordinal automata [Büchi65].
Complexity improved using a complicated coding by finite words [Maurin97].

"We have a proof without theorem" Wolfgang Thomas, 2002
(closure of tree automata under boolean operations and projection)

Presburger arithmetic: $\mathrm{FO}(\mathbb{N}, +)$

[Presburger1930, Skolem1931]: quantifier elimination

[Büchi1960]: using finite automata

integer $x \in \mathbb{N}$	$\leftrightarrow$	$u \in \{0,1\}^*$ in binary
relation $x + y = z$	$\leftrightarrow$	automaton over $\{0,1\}^3$
models: set $X \subseteq \mathbb{N}$	$\leftrightarrow$	automaton over $\{0,1\}$
formula $\psi(x_1, \ldots, x_k)$	$\leftrightarrow$	automaton over $\{0,1\}^k$
quantification $\exists$	$\leftrightarrow$	projection ($\leadsto$ non-determinism)
$\vee$, $\wedge$	$\leftrightarrow$	union, product
negation	$\leftrightarrow$	complementation ($\leadsto$ costly)

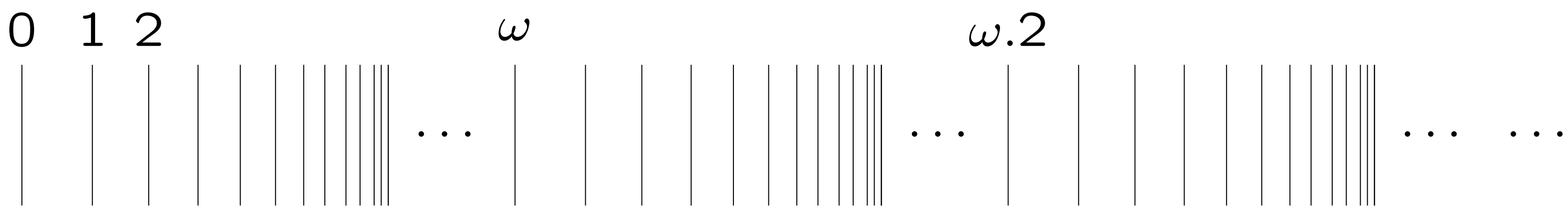
Induction on the structure of the formula.

Emptiness is decidable

see [Klaedtke04]

Ordinal arithmetic

$1 + \omega = \omega \neq \omega + 1 \qquad 2.\omega = \omega \neq \omega.2$



$\forall$ ordinal α: $\alpha = \{\beta : \beta < \alpha\}$

$$\begin{aligned} \omega^i + \omega^j &= \omega^j && \text{if } i < j \\ &= \omega^j.2 && \text{if } i = j \end{aligned}$$

Cantor Normal Form

$\forall\ 0 < \alpha < \omega^\omega$, $\exists$ unique integers $p, n_0, n_1, \ldots, n_p$, s.t.

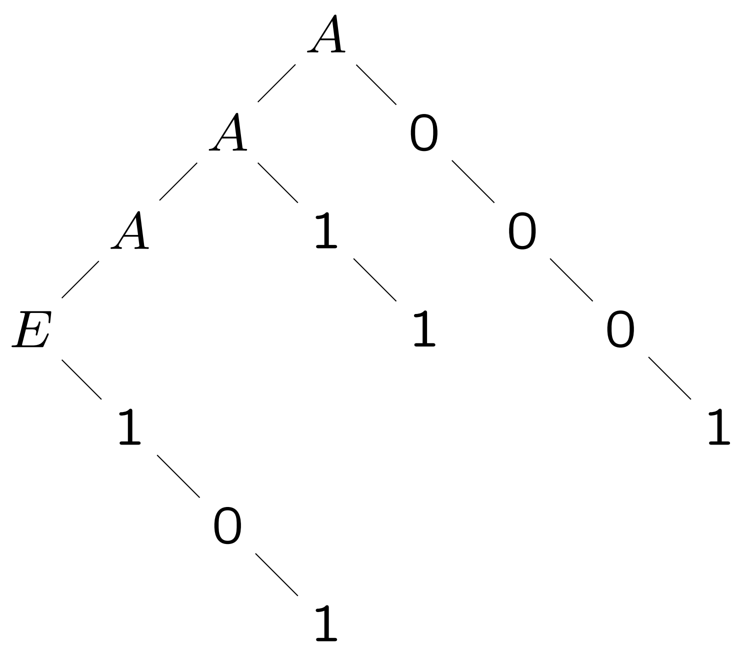
$$n_p > 0 \text{ and } \alpha = \omega^p n_p + \omega^{p-1} n_{p-1} + \ldots + \omega^1 n_1 + n_0$$

Given $\alpha = \omega^p n_p + \ldots + \omega^1 n_1 + n_0$
and $\alpha' = \omega^{p'} n'_{p'} + \ldots + \omega^1 n'_1 + n'_0$ in CNF,

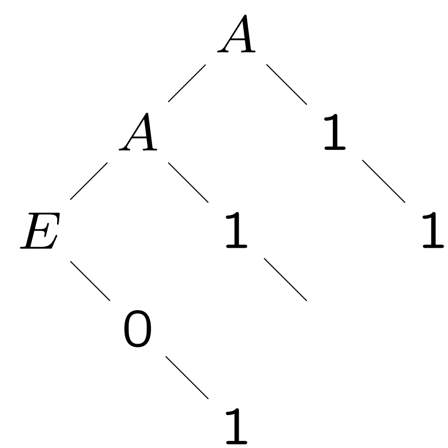
$$\alpha + \alpha' = \omega^p n_p + \ldots + \omega^{p'} (n_{p'} + n'_{p'}) + \ldots + \omega^1 n'_1 + n'_0$$

Representation by finite trees

$\omega^3.5 + \omega.3 + 8$



$\omega^2.2 + \omega.1 + 3$



addition	$\leftrightarrow$	tree-automaton over $\{E, A, 0, 1\}^3$
boolean operations	$\leftrightarrow$	closure properties
$\exists$ quantification	$\leftrightarrow$	projection

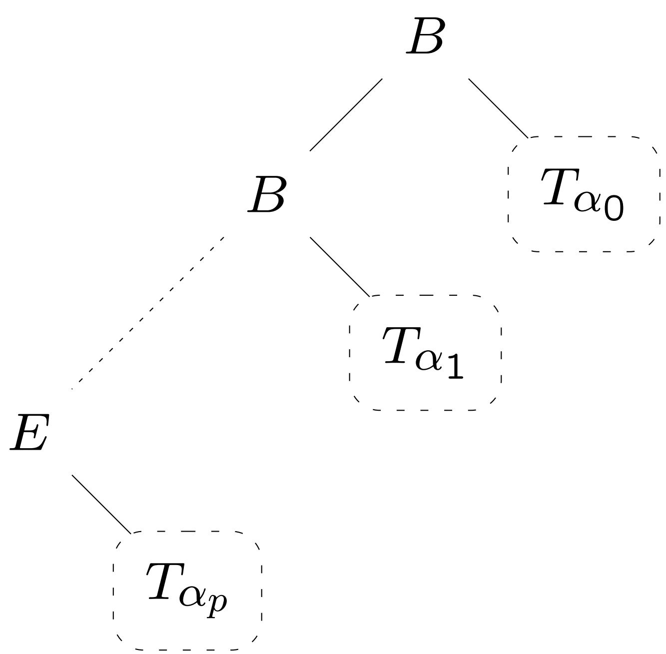
Induction on the formula. Emptiness decidable

Add dummy symbols #

Extension to ω^{ω^i}

Cantor Normal Form $\rightsquigarrow$ Any $\beta < \omega^{\omega^2}$ is uniquely written as

$$\omega^{\omega.p}\alpha_p + \ldots + \omega^{\omega.2}\alpha_2 + \omega^{\omega}\alpha_1 + \alpha_0 \text{ , where } p < \omega, \alpha_i < \omega^{\omega}, \alpha_p > 0 \text{ .}$$



and so on by induction . . .

Complexity

Let $\mathsf{Tower}(0,n) = n$ and $\mathsf{Tower}(k+1,n) = 2^{\mathsf{Tower}(k,n)}$.

Thm: $\forall i < \omega \; \exists c_i$ such that $\mathrm{FO}(\omega^{\omega^i}, +)$ is decidable in time $\mathcal{O}(\mathsf{Tower}(n, c_i))$, where n is the length of the formula.

Best known result:
via reduction to Weak Monadic Second Order logic $(\omega^\omega, <)$,
in turn decidable in time $\mathcal{O}(\mathsf{Tower}(6n, c'))$ [Maurin96].

lower bound?

Monadic Second Order $(\mathbb{N}, <)$ [Büchi62]

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	$\cdots$
$X \subseteq \mathbb{N}$	$\bullet$		$\bullet$	$\bullet$			$\bullet$		$\bullet$	$\bullet$	$\cdots$

set $X \subseteq \mathbb{N}$			$\leftrightarrow$	infinite word $v \in \{0,1\}^\omega$
$x \in \mathbb{N}$	$\leftrightarrow$	$\{x\} \subseteq \mathbb{N}$	$\leftrightarrow$	$0^x 1 0^\omega$
$x \in X$			$\leftrightarrow$	Büchi automaton over $\{0,1\}^2$
$x < y$			$\leftrightarrow$	Büchi automaton over $\{0,1\}^2$
formula $\psi(x_1, \ldots, x_k)$			$\leftrightarrow$	Büchi automaton over $\{0,1\}^k$
quantification $\exists$			$\leftrightarrow$	projection ($\rightsquigarrow$ non-determinism)
$\vee$, $\wedge$			$\leftrightarrow$	union, product
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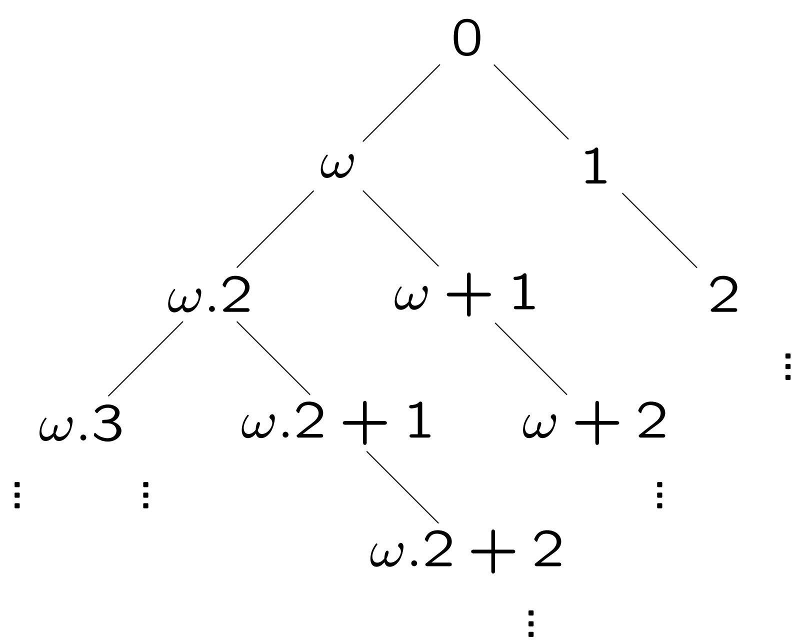
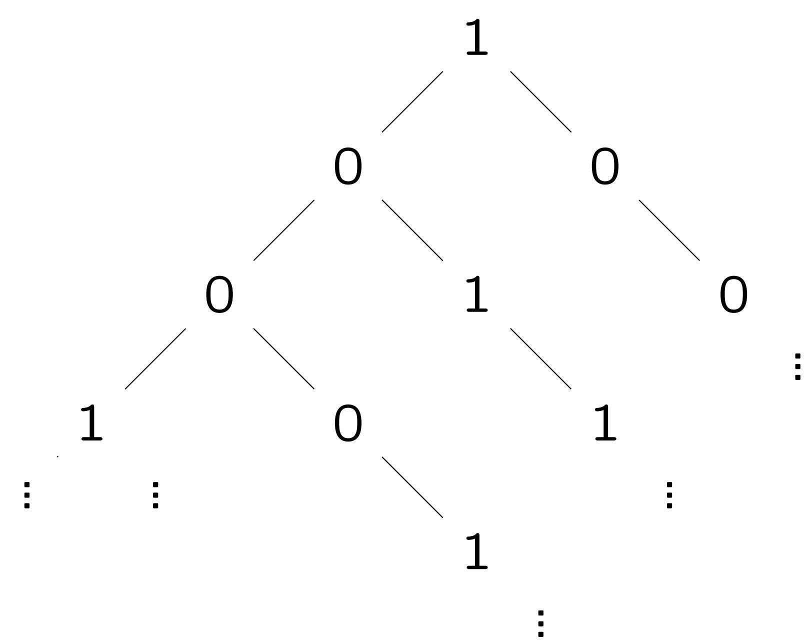
Non-elementary lower bound

Representation of any $X \subseteq \omega^2$ by an infinite tree

positions



labels $\in \{0, 1\}$



$\{0, \omega + 1, \omega + 2, \omega.2 + 2, \omega.3\}$

Use Rabin tree-automaton (or Muller or parity)
cannot be determinized yet complemented

Example of automata

Test X is $\{x\}$: non-deterministicaly search the unique node labeled 1

Test $x < y$: either on the same branch, or on different branches . . .

Other cases are easy and similar to $(\mathbb{N}, <)$

Extension to ω^i: as in the case of FO$(\omega^{\omega^i}, +)$ with e.g.

$$X \subseteq \omega^3 \text{ is written } \bigcup_{j<\omega} \left(\omega^2.j + X_j\right) \text{ where } \forall j : X_j \subseteq \omega^2$$

Thm: $\forall i < \omega\ \exists d_i$ such that MSO$(\omega^i, <)$ is decidable in time $\mathcal{O}(\text{Tower}(n, d_i))$, where n is the length of the formula.

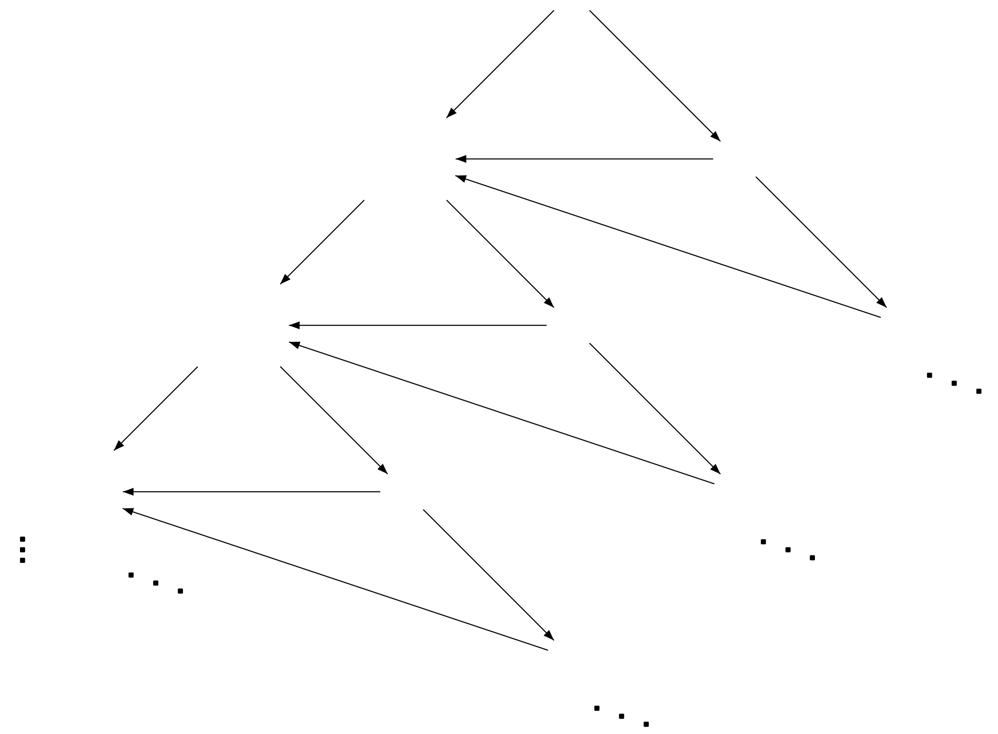
Lower bound in Tower(n, c) even for $i = 1$

[Maurin96] WeakMSO $\mathcal{O}(\text{Tower}(6n, d'))$

MSO interpretation

The graph of $(\omega^i, <)$ is prefix recognizable, *ie* at level 1 in the Caucal Hierarchy [Caucal02]

Transitive closure of



Open problem: higher ordinals at higher levels of the hierarchy?

Weak MSO and FO

Weak MSO: same syntax as MSO, but sets are interpreted as finite

Any ordinal β is uniquely written as

$$2^{\gamma_{n-1}} + \ldots + 2^{\gamma_0} \text{ where } (\gamma_{n-1} > \gamma_{n-2} > \ldots > \gamma_0)$$

Examples: $2^\omega = \omega$, $2^{\omega.i+j} = 2^{\omega.i}.2^j = \omega^i.2^j$, $2^{\omega^2} = (2^\omega)^\omega = \omega^\omega$.

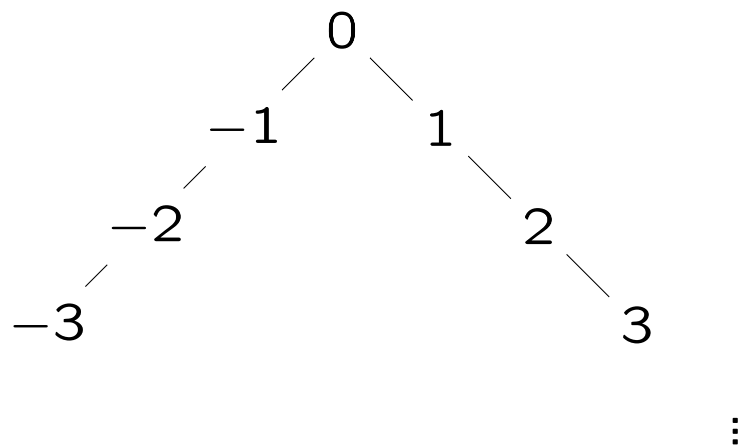
[Buechi65] theories WMSO($\alpha, <$) and FO($2^\alpha, +, E$) are equireducible in linear time. $\rightsquigarrow$ WMSO($\omega^{i+1}, <$) and FO($\omega^{\omega^i}, +, E$)

Any ordinal β can yet be written in a unique way in the form

$$\alpha = \gamma.\omega^\omega + \omega^p n_p + \omega^{p-1} n_{p-1} + \ldots + \omega^1 n_1 + n_0 \quad \text{where} \quad n_p > 0$$

$WMSO(\alpha, <)$ depends only on $((\gamma > 0), n_p, \ldots, n_0)$

Extension to other linear orderings



Consider $-\omega + \omega \quad (-\omega).\omega \quad \ldots$

Conjecture: new proof of [CartonRispal04]

Open problems: linear orders of infinite rank, higher ordinals