Théorie monadique des représentations . . .

David Janin (Bordeaux)

joint work with Anuj Dawar (Cambridge)

Réunion ACI Sécurité (VERSYDIS)

Théorie monadique des représentations finies de mots ultimement périodiques

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Models and Logic

Models: directed Σ-labeled graph with distinguished source

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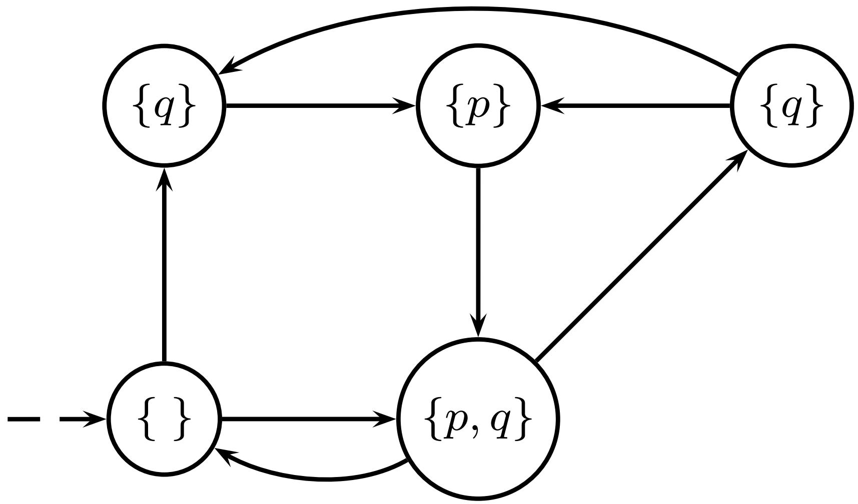

Models: directed Σ-labeled graph with distinguished source

$$M = \langle V, E \subseteq V \times V, s \in V, \lambda : V \to \Sigma = \mathcal{P}(\textit{\textbf{Prop}}) \rangle$$

Logic: monadic second order logic on those graphs on signature $\{E\}$ (edges) and ***Prop*** (unary predicates).

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Partial answers and other questions:

- At least all mu-calculus definable properties !
- Other PTime definable classes of graphs ?
- NPcomplete classes of graphs ?

Known result in the finite

Theorem [Rosen 90's]: The bisimulation invariant fragment of FO on the finite equals modal logic.

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Question 2: What about bisimulation invariant monadic Σ_1 on finite unary graphs ?

Unary graphs or lassos

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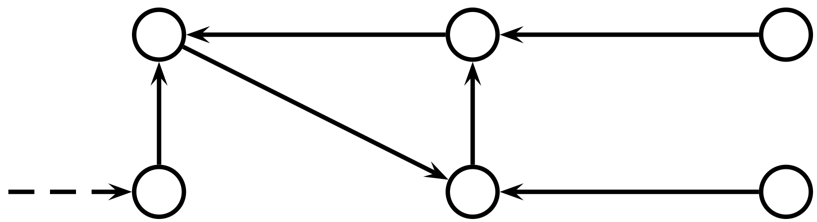

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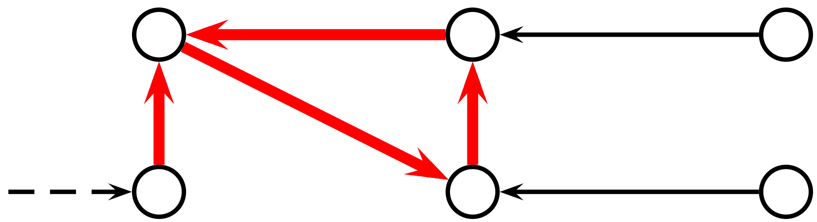

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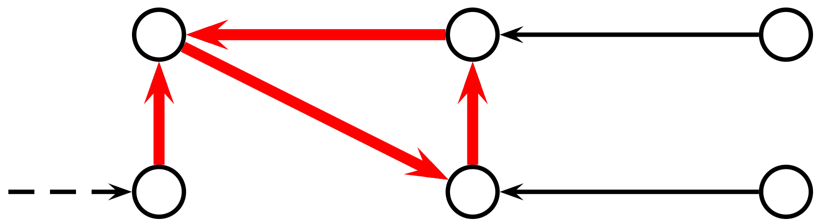

Def.: Lasso : unary graph where every vertex is reachable from the root and the root has no predecessor.

Simple facts on lassos

Facts:

- Every finite unary graph M is bisimilar to a lasso (i.e. induced by the set of reachable vertices. . .).

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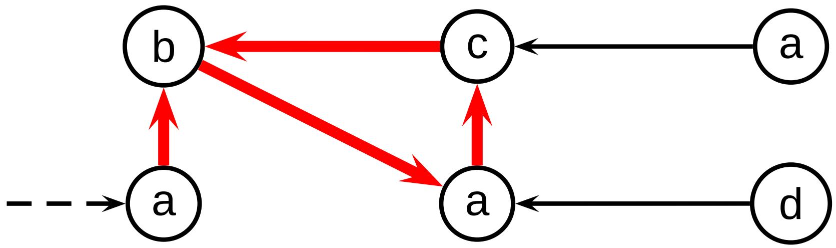

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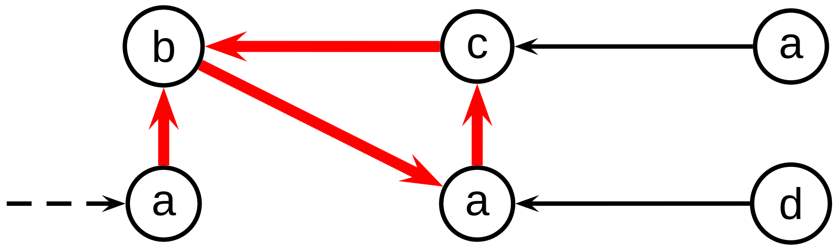

A finite word to the cycle: $u = a$.
A finite word on the cycle: $v = bac$.
A inf. encoded word: $w = a.(bac)^\omega$.

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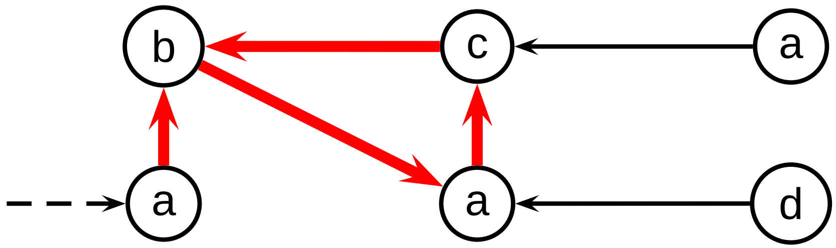

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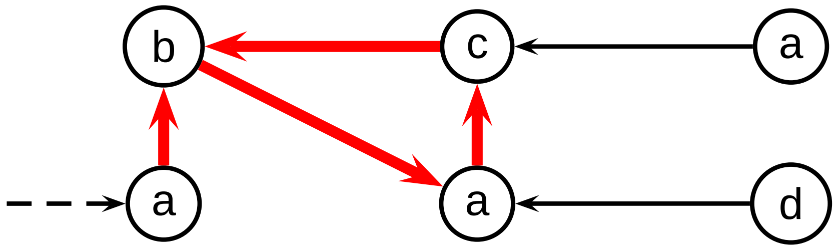

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- Write $M_{u,v}$ for such a lassos.

The simple problem

Def.: Let φ be an MS-formula on unary graphs and let

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Question: Other properties ?

Main result

Def.: For $L \subseteq \Sigma^\omega$, let $kern(L) = \{u.v^\omega \in L : u \in \Sigma^+, v \in \Sigma^+\}$.

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Theorem: For any bis. inv. formula φ on unary graph:

- L_φ is the kernel of an ω-regular language,
- φ is equivalent to a monadic Σ_1 formula.

Reg. to monadic Σ_1 (1)

Lemma 1: For every regular ω-language L there is a bis. invariant monadic Σ_1 formula φ such that $kern(L) = L_\varphi$

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- Take a normalized non deterministic Buchi automaton $\mathcal{A}$ that recognizes L, i.e. such that $u.v^\omega \in L$ if and only if there is an accepting state q s.t.

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Corollary: on finite unary graphs, $NC_1 \subset$ bis.inv. Σ_1 and the inclusion is strict !

Lemma 2: For any bis. inv. MS-formula φ on unary graphs there is an ω-regular language L such that

$$kern(L) = L_\varphi$$

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Lemma: There is a finite collection of regular languages $\{U_i, V_i\}_{i\in I}$ such that:

$$M_{u,v} \models \varphi \text{ iff } u \in U_i \text{ and } v \in V_i \text{ for some } i \in I$$

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Conversely ? If $w \in U_i.V_i^\omega$ ultimately periodic we have:

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And then ...?

Lemma: For each $M_{u,v}$, there a triple $(j,r,s) \in I \times \Sigma^+ \times \Sigma^+$ such that:

- $u.v^\omega = r.s^\omega$, i.e.
- for all k, $r.s^k \in U_j$ and $s.s^k \in V_j$.

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And both u_j and V_j are regular...

Def.: for such a special triple $t = (j, r, s)$ let

$$D_t = [r]_{U_j}.([s]_{U_j} \cap [s]_{V_j}) \text{ and } E_t = ([s]_{U_j} \cap [s]_{V_j})$$

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Lemma:

- $D_t \subseteq U_j$, $E_t \subseteq V_j$ hence for any u and $v \in \Sigma^+$, if $u \in D_t$ and $v \in E_t$ then $M_{u,v} \models \varphi$,
- $D_t.E_t^+ \subseteq D_t$ and $E_t^+ \subseteq E_t$.

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$\subseteq$. By construction of D_js and E_js...

$\supseteq$.

- Let $w \in D_t.E_t^\omega$ for some special triple $t = (j, r, s)$ with w ultim. per. We must show that $w \in L_\varphi$, i.e. decompose w in $u.v^\omega$ with $M_{u,v} \models \varphi$.

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- Take $u = u_1.v_1.\cdots.v_n$ in D_t (by prop. of D_t) and $v = v_{n+1}.\cdots.v_m$ in E_t (by prop. of E_t) so that $M_{u,v} \models \varphi$.

Infinite vs finite in unary case ?

Remark: On arbitrary finite and inf. unary models bis. inv. mon Σ_1 defined closed regular languages while on finite models it defines all ω-regular languages.

Theorem: On unary graphs the following formalisms are expressively equivalent:

- (bis. inv.) monadic Σ_1,
- (bis. inv.) MSO,
- μ-calculus.