Alternating Timed Automata

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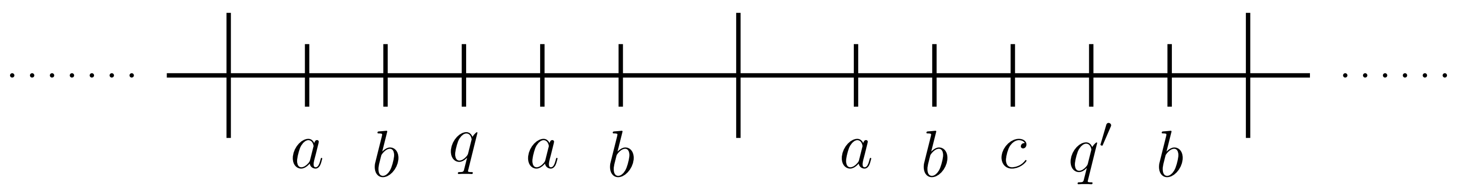
- A model of alternating timed automata.
- Decidable emptiness problem.
- But very high complexity.

Undecidability of the universality problem

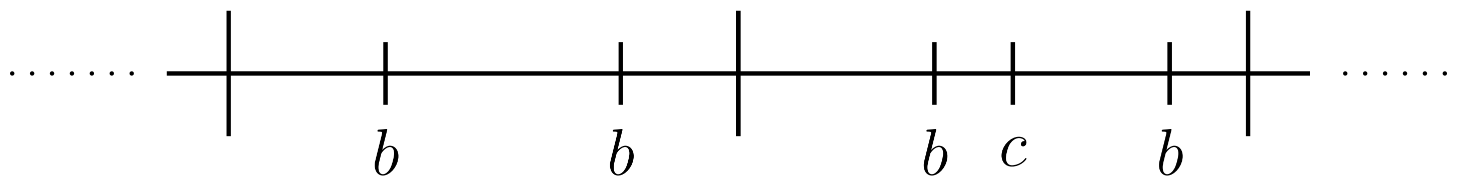
Thm [Alur & Dill]: The universality problem for timed automata is undecidable.



Cor: The emptiness problem for alternating timed automata will be undecidable.

One clock restriction

Thm [Ouaknine and Worrell]: The universality problem for automata with one clock is decidable.



Universality $\mapsto$ accessibility in a well-structured system

Question: What is the complexity of this problem?

Our contribution

- Definition of alternating timed automaton (ATA).
- Decidability of the emptiness problem for one-clock ATA (small extension of O & W).
- Non primitive recursive lower bound for the problem (reduction of accessibility in lossy channel systems).
- Undecidability for the extension of one-clock ATA with ε-transitions.

Alternating timed automaton

- Clock variables $\mathcal{C}$ and clock constrains $\Psi(\mathcal{C})$:

$$\sigma \quad ::= \quad x < c \mid x \leq c \mid \sigma_1 \wedge \sigma_2 \mid \neg\sigma.$$

- Transition relation as a partial function:

$$Q \times \Sigma \times \Phi(\mathcal{C}) \dot{\to} \mathcal{P}(Q \times \mathcal{P}(\mathcal{C})).$$

- Alternating timed automaton:

$$\mathcal{A} = (Q,\ q_0,\ \Sigma,\ \mathcal{C},\ \delta,\ F)$$

where

$$\delta : Q \times \Sigma \times \Phi(\mathcal{C}) \dot{\to} \mathcal{B}^+(Q \times \mathcal{P}(\mathcal{C})).$$

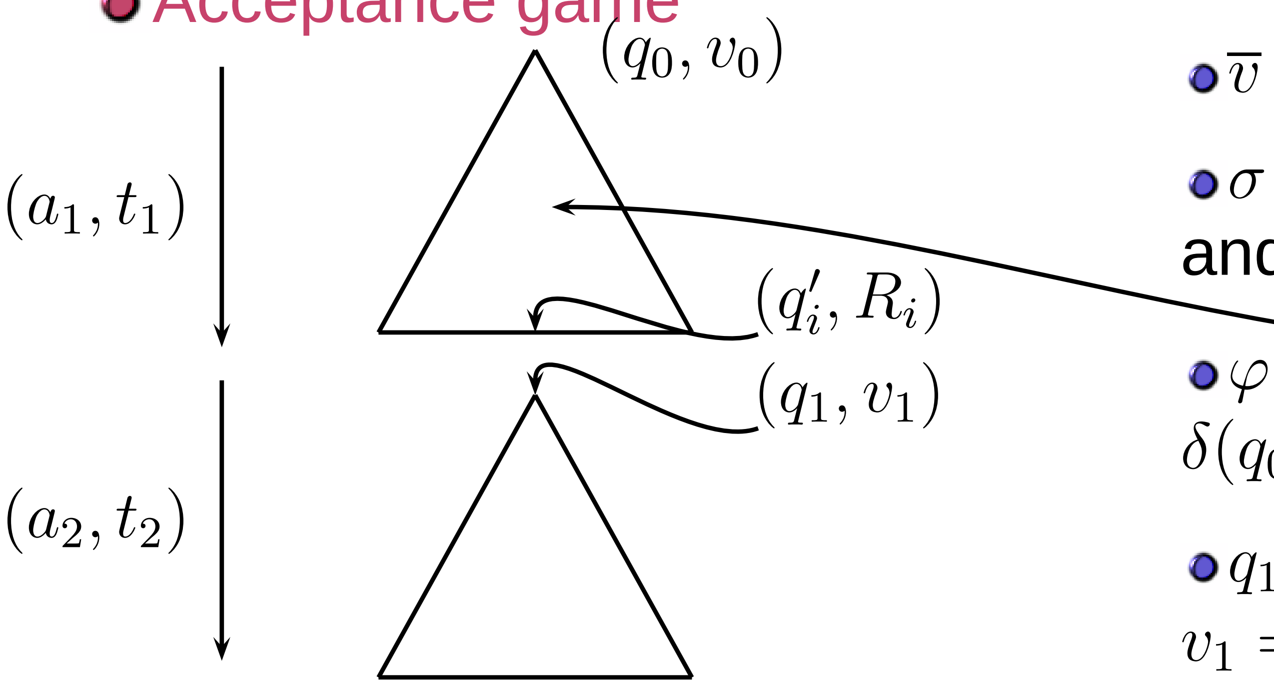
- (Partition) For every q and a, the set $\{[\sigma] : \delta(q,a,\sigma) \text{ is defined}\}$ gives a (finite) partition of $(\mathbb{R}_+)^{\mathcal{C}}$.

Semantics

- Timed word: $w = (a_1, t_1)(a_2, t_2) \dots (a_n, t_n)$

  Example: $(a, 0.5)(b, 1.3)(c, 2.5)$

- Acceptance game

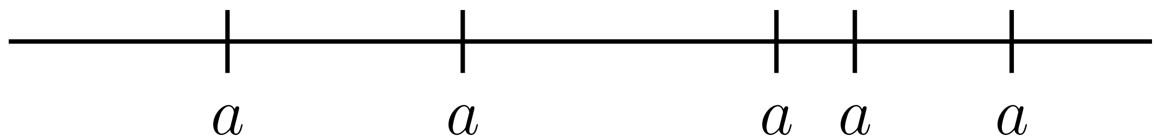


  - $\overline{v} = v_0 + t_1$;
  - σ unique s.t $\sigma \models \overline{v}$ and $\delta(q_0, a_1, \sigma)$ defined;
  - $\varphi((q_1', R_1), \dots, (q_n', R_n)) = \delta(q_0, a_1, \sigma)$
  - $q_1 = q_i'$ and $v_1 = \overline{v}[R_i := 0]$.

- Automaton wins if it gets to an accepting state after (a_n, t_n).

Example

- L = words w with no two a's at distance 1



- The alternating automaton for L:

$$q_0, a, \mathbf{tt} \mapsto (q_0, \emptyset) \wedge (q_1, \{x\})$$

$$q_1, a, x \neq 1 \mapsto (q_1, \emptyset) \qquad q_1, a, x = 1 \mapsto (\bot, \emptyset)$$

- Accepting states: q_0, q_1

Closure properties

Rem: ATA languages are effectively closed on boolean operations. The constructions do not increase the number of clocks.

Def: Automaton is purely existential if there is no conjunction in the transition rules. It is purely universal if no disjunction.

Rem: Every standard nondeterministic automaton is equivalent to a purely-existential one.

Each automaton can be converted to the one satisfying (Partition) condition.

Problems of interest

- We are interested in the:
  emptiness, universality and inclusion ($L(\mathcal{A}) \subseteq L(\mathcal{B})$) problems.

Rem: All these problems are undecidable.

Thm [Ouaknine and Worrell]: The universality problem for automata with one clock is decidable.

Proviso: In what follows we restrict to automata with only one clock.

Rem: The power of ATA with one clock is incomparable with that of timed automata.

- Language with b at times $t_1 < t_2 < t_1{+}1 < t_2{+}1$ with no b at time between t_1 and t_2 and a b between $t_1{+}1$ and $t_2{+}1$.

The emptiness problem

Thm: Emptiness is decidable for one-clock ATA.

Cor: Containment is decidable for one-clock ATA.

Proof:

- Power-set construction as for ordinary automata:

  macro-states = sets $\{(q_1, v_1), \ldots, (q_n, v_n)\}$ of configurations
  Transition sytem $\mathcal{T}$ with macro states and transitions given by $\mathcal{A}$

$$\{(q_1, v_1), \ldots, (q_n, v_n)\}$$

$$\{(q_{1,1}, v_{1,1}), \ldots, (q_{1,k}, v_{1,k}), \ldots, (q_{n,1}, v_{n1}), \ldots, (q_{n,k}, v_{n,k})\}$$

- Bad macro-state if all of the states accepting.
- $\mathcal{A}$ accepts a word iff there is a path to a bad state.
- Non-emptiness $\equiv$ reachability of a bad state in $\mathcal{T}$.

Proof cont.

- Representing a macro-state $\{(q_1, v_1), \ldots, (q_n, v_n)\}$

$\texttt{reg} := \{\{0\}, (0,1), \{1\}, (1,2), \ldots, (c_{\text{max}}-1, c_{\text{max}}), \{c_{\text{max}}\}, (c_{\text{max}}, +\infty)\}$.

- Define $H(\{(q_1, v_1), \ldots, (q_n, v_n)\}) \in (\mathcal{P}(Q \times \texttt{reg}))*$

(1) Replace each (q, v) by $(q, \texttt{reg}(v), \texttt{fract}(v))$.

(2) Sort the triples w.r.t. $\texttt{fract}(v)$.

(3) Group triples having the same $\texttt{fract}(v)$.

(4) Forget about $\texttt{fract}(v)$.

Fact: Define $P \sim P'$ if $H(P) = H(P')$. This is a bisimulation rel.
Cor: $L(\mathcal{A}) \neq \emptyset$ iff a bad state is reachable in $\mathcal{H} = H(\mathcal{T})$.

$\mathcal{H}$ is a well structured system

- Let $a_1 \dots a_n \preccurlyeq b_1 \dots b_m$ if there is $f : [n] \to [m]$ such that

$$a_i \subseteq b_{f(i)}$$

- Relation $\preccurlyeq$ is a well-order.

Lemma: If $W_1 \preccurlyeq W_2$ and $W_2 \to W_2'$ then there is W_1' with $W_1 \to W_1'$ and $W_1' \preccurlyeq W_2'$.

Lemma: If $W \preccurlyeq W'$ and W' is bad then so is W.

Lemma: It is decidable if a bad state is reachable in $\mathcal{H}$.

The lower bound

Thm: The complexity of the emptiness problem for one-clock purely-universal ATA is not bounded by a primitive recursive function.

Cor: The complexity of the universality problem for one-clock purely existential alternating (i.e., nondeterministic) timed automata is not bounded by a primitive recursive function.

Proof: Reduction of reachability in lossy channel systems.

Lossy channel systems

- A channel system is $\mathcal{S} = \langle Q, q_0, \Sigma, \Delta \rangle$ where:

$$\Delta \subseteq Q \times (\{!a : a \in \Sigma\} \cup \{?a : a \in \Sigma\} \cup \{\varepsilon\}) \times Q$$

- $qw \xrightarrow{(q,\varepsilon,q')} q'w$
- $qw \xrightarrow{(q,!a,q')} q'aw$
- $qwa \xrightarrow{(q,?a,q')} q'w$
- Loss of messages bush $\sqsubseteq$ `abrisousroche`
- Lossy computation step:

$$qw \qquad qw_1 \xrightarrow{\gamma} q'w'_1 \qquad q'w_2$$

$$w \sqsupseteq w_1 \qquad w'_1 \sqsupseteq w_2$$

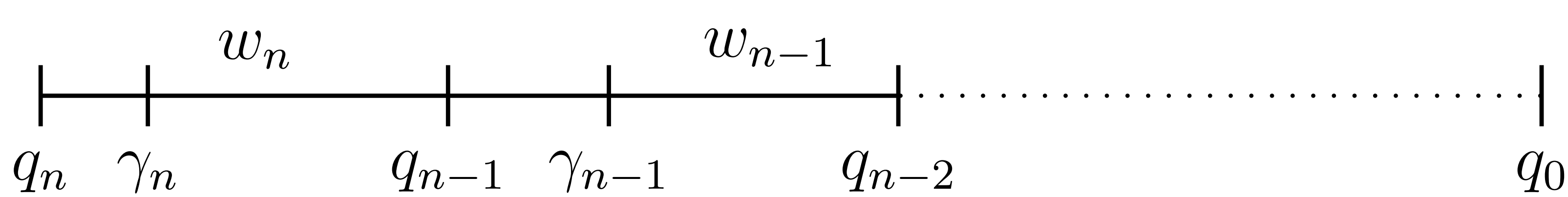
Lossy reachability

$$(q_0, \epsilon) \xrightarrow{\gamma_1} (q_1, w_1) \xrightarrow{\gamma_2} (q_2, w_2) \quad \dots \quad \xrightarrow{\gamma_n} (q_n, w_n)$$

- Lossy reachability problem: Given a system $\mathcal{S}$ and a configuration $q_f w_f$ decide if there is a computation ending in $q_f w_f$.

Thm [Schnoebelen]: The lossy reachability problem for channel systems has nonprimitive recursive complexity.

- Representation of a computation as a timed word:



Computations as timed words

$$(q_0, \epsilon) \xrightarrow{\gamma_1} (q_1, w_1) \xrightarrow{\gamma_2} (q_2, w_2) \quad \ldots \quad \xrightarrow{\gamma_n} (q_n, w_n)$$

$$(q_n, t_n)(\gamma_n, t'_n)v_n \; (q_{n-1}, t_{n-1})(\gamma_{n-1}, t'_{n-1})v_{n-1} \quad \ldots \quad (q_1, t_1)(\gamma_1, t'_1)v_1 \; (q_0, t$$

(P1) Structure

(P2) Distribution in time: $t_i = n - i$.

(P3a) Epsilon move: if $\gamma_i = (q, \varepsilon, q')$ then $(v_i + 1) \sqsubseteq v_{i-1}$.

(P3b) Write move: . . .

(P3c) Read move: . . .

Verifying the encoding

- The step for the ε move:

$$s_{\text{step}}, \langle q, \varepsilon, q' \rangle, \mathbf{tt} \mapsto s_{\text{channel}}$$

$$\begin{aligned} s_{\text{channel}}, a, \mathbf{tt} &\mapsto s_{\text{channel}} \wedge (s_a^{+1}, \{x\}), \quad \text{for } a \in \Sigma \\ s_{\text{channel}}, Q, \mathbf{tt} &\mapsto \top \end{aligned}$$

$$\begin{aligned} s_a^{+1}, a, x = 1 &\mapsto \top \\ s_a^{+1}, \overline{\Sigma}, x < 1 &\mapsto s_a^{+1} \end{aligned}$$

- This way for a given channel system $\mathcal{S}$ and a configuration $q_f w_f$ we construct an ATA $\mathcal{A}$ accepting the encodings of lossy computations of $\mathcal{S}$ ending in $q_f w_f$.

Silent transitions

- It is not obvious how to introduce ε-transitions.
  (Should we allow uncountable number of copies running?)

- A nondeterministic timed automaton with ε-transitions over Σ is a nondeterministic timed automaton over $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$.

- A timed word over Σ is accepted by a timed automaton $\mathcal{A}$ with ε-transitions if there is a timed word v over Σ_ε accepted by $\mathcal{A}_\varepsilon$ such that $w = v\downarrow_\varepsilon$.

Thm: The universality problem for one-clock nondeterministic timed automata with ε-transitions is undecidable.

Proof: Encoding of computations of non-lossy computations of channel systems by universal automata.

A loss of a message can be checked with ε-transitions.

To do

- Adding event-clocks to the model.
- Logical characterizations of ATA languages.
- Different syntax to improve the complexity.
- Infinite words.