A simple model of trees for unicellular maps

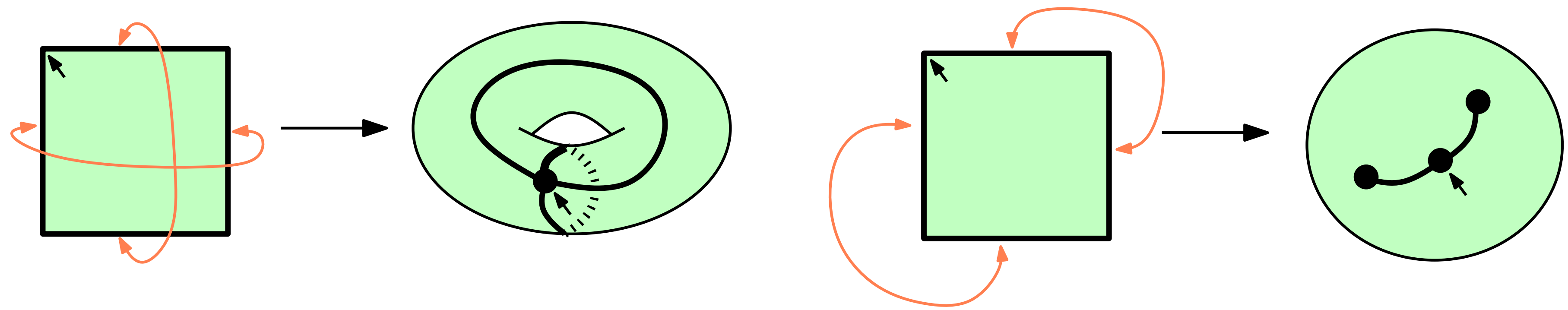
Guillaume Chapuy (LIAFA, Paris-VII)

joint work with
Valentin Féray (LaBRI, Bordeaux-I)
Éric Fusy (LIX, Polytechnique)

FPSAC, August 2012, Nagoya, Japan

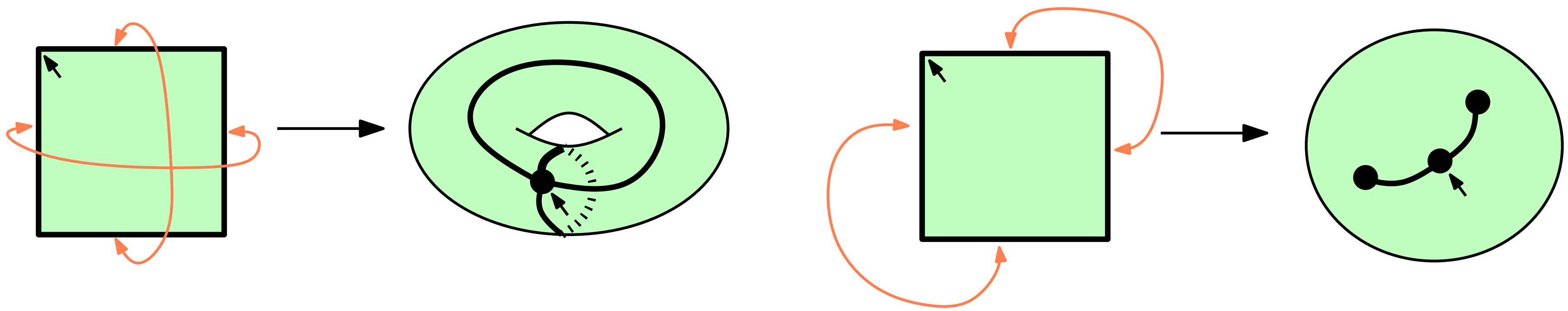
Unicellular maps (a.k.a. “one-face” maps)

- Start with a (rooted) $2n$-gon, and paste the edges pairwise in order to form an orientable surface.



Unicellular maps (a.k.a. "one-face" maps)

• Start with a (rooted) $2n$-gon, and paste the edges pairwise in order to form an orientable surface.

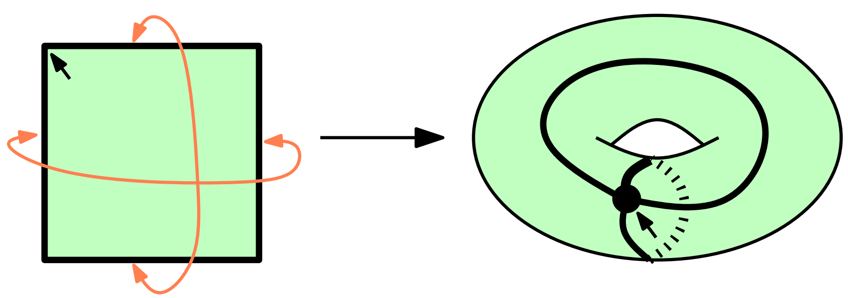


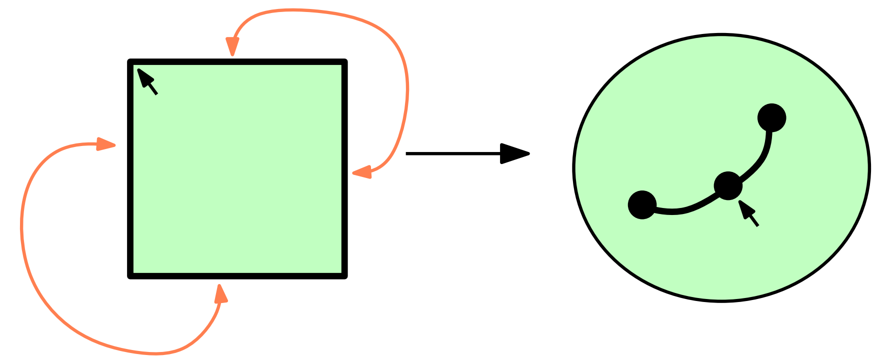
• One obtains an n-edge graph drawn on the surface. The number of handles (=genus) of the surface is given combinatorially by Euler's formula:

$$\#\text{vertices} = n + 1 - 2g$$

Unicellular maps (a.k.a. "one-face" maps)

- Start with a (rooted) $2n$-gon, and paste the edges pairwise in order to form an orientable surface.



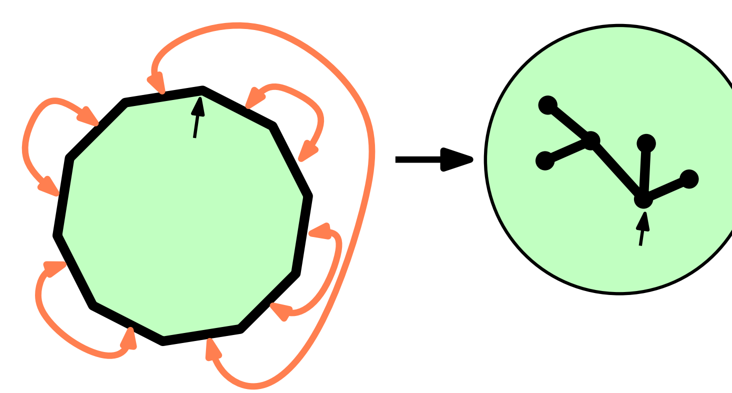


- One obtains an n-edge graph drawn on the surface. The number of handles (=genus) of the surface is given combinatorially by Euler's formula:

$$\#\text{vertices} = n + 1 - 2g$$

- The number of unicellular maps of size n is $(2n-1)!!$

- What if we fix the genus ? For example, on the sphere (genus 0), unicellular maps $=$ plane trees... so there are $\mathrm{Cat}(n)$ of them.



Unicellular maps: counting!

- Let $\epsilon_g(n)$ be the number of unicellular maps with n edges and genus g.

- Are these numbers interesting ? Yes!

$$\epsilon_0(n) = \mathrm{Cat}(n)$$
$$\epsilon_1(n) = \frac{(n+1)n(n-1)}{12}\mathrm{Cat}(n)$$
$$\epsilon_2(n) = \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440}\mathrm{Cat}(n)$$
$$\ldots$$

- These numbers are connection coefficients in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ (all map numbers are - but this is not really the point of this talk).

Unicellular maps: some chosen formulas

[Lehman-Walsh 72]

$$\epsilon_g(n) = \left(\sum_{\gamma \vdash g} \frac{(n+1-2g)_{2\ell(\gamma)+1}}{2^{2g} \prod_i m_i!(2i+1)^{m_i}} \right) \mathrm{Cat}(n)$$

no bijective proof!

[Harer-Zagier 86]
(summation form)

$$\sum_{g \geq 0} \epsilon_g(n) y^{n+1-2g} = (2n-1)!! \sum_{i \geq 1} 2^{i-1} \binom{n}{i-1} \binom{y}{i}$$

nice bijective proof [Bernardi10] building on [Lass 01, Goulden Nica 05]

[Harer-Zagier 86] (recurrence form)

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$

no bijective proof! (except $g=0$ Rémy's bijection)

... and many others! [Jackson 88, Goulden-Jackson 92, Goupil-Schaeffer 98, Schaeffer-Vassilieva 08, Morales-Vassilieva 09, Ch. 09, Bernardi-Ch. 10, ...].

Unicellular maps: some chosen formulas

[Lehman-Walsh 72]

$$\epsilon_g(n) = \left(\sum_{\gamma \vdash g} \frac{(n+1-2g)_{2\ell(\gamma)+1}}{2^{2g} \prod_i m_i!(2i+1)^{m_i}} \right) \mathrm{Cat}(n)$$

no bijective proof!

[Harer-Zagier 86] (summation form)

$$\sum_{g \geq 0} \epsilon_g(n) y^{n+1-2g} = (2n-1)!! \sum_{i \geq 1} 2^{i-1} \binom{n}{i-1} \binom{y}{i}$$

nice bijective proof [Bernardi10] building on [Lass 01, Goulden Nica 05]

[Harer-Zagier 86] (recurrence form)

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$

no bijective proof! (except $g = 0$ Rémy's bijection)

... and many others! [Jackson 88, Goulden-Jackson 92, Goupil-Schaeffer 98, Schaeffer-Vassilieva 08, Morales-Vassilieva 09, Ch. 09, Bernardi-Ch. 10, ...].

[Goupil-Schaeffer 98] for $\lambda \vdash 2n$, $\lambda = 1^{m_1} 2^{m_2} \ldots$:

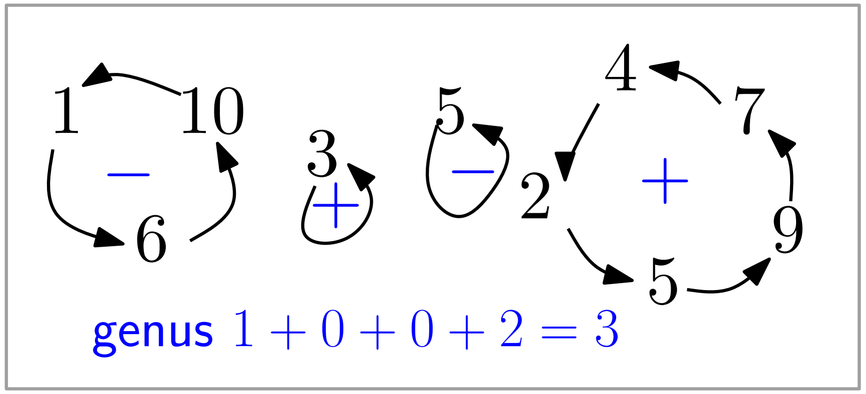
$$\epsilon_g(n; \lambda) = \frac{(l+2g-1)!}{2^{2g-1} \prod_i m_i!} \sum_{\gamma_1+\gamma_2+\cdots+\gamma_l = g} \prod_i \frac{1}{2\gamma_i+1} \binom{\lambda_i - 1}{2\gamma_i}$$

vertex degrees

no bijective proof!

Our result, in short

- A C-permutation of a set S:

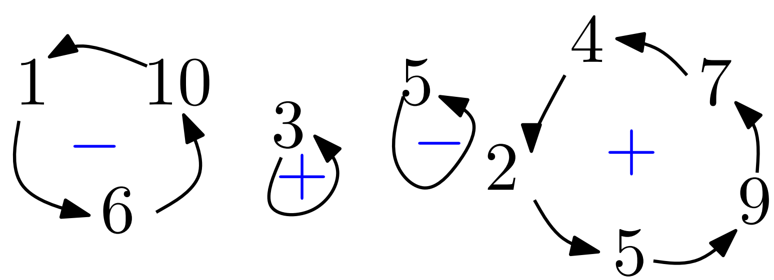


- all cycles have odd length
- each cycle carries a sign in $\{+,-\}$
- its genus is $g := \sum_i k_i$ where $(2k_i+1)$ are the cycle-lengths

$$\#\text{cycles} = |S| - 2g$$

Our result, in short

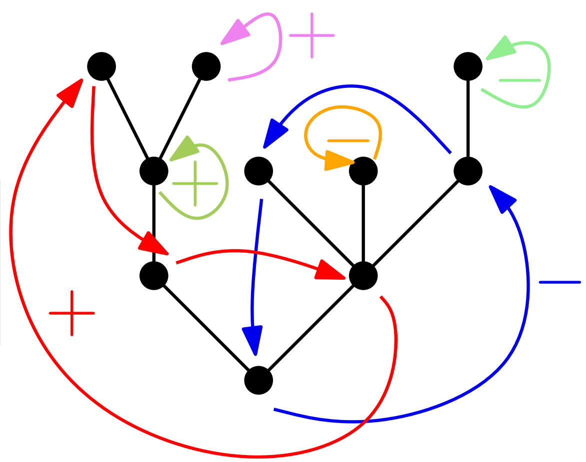
- A C-permutation of a set S:
  - all cycles have odd length
  - each cycle carries a sign in $\{+,-\}$
  - its genus is $g := \sum_i k_i$ where $(2k_i+1)$ are the cycle-lengths



genus $1+0+0+2=3$

$$\#\text{cycles} = |S| - 2g$$

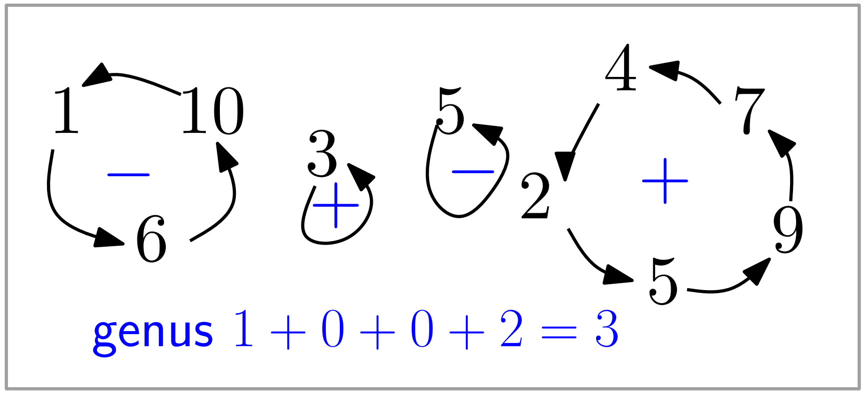
- C-decorated tree = a rooted plane tree equipped with a C-permutation of its vertices:



a C-decorated tree of genus 2

Our result, in short

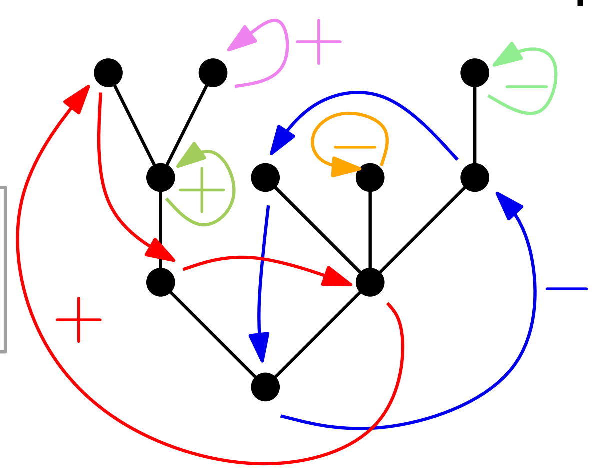
- A C-permutation of a set S:
  - all cycles have odd length
  - each cycle carries a sign in $\{+,-\}$
  - its genus is $g := \sum_i k_i$ where $(2k_i+1)$ are the cycle-lengths



$$\#\text{cycles} = |S| - 2g$$

- C-decorated tree = a rooted plane tree equipped with a C-permutation of its vertices:

a C-decorated tree of genus 2

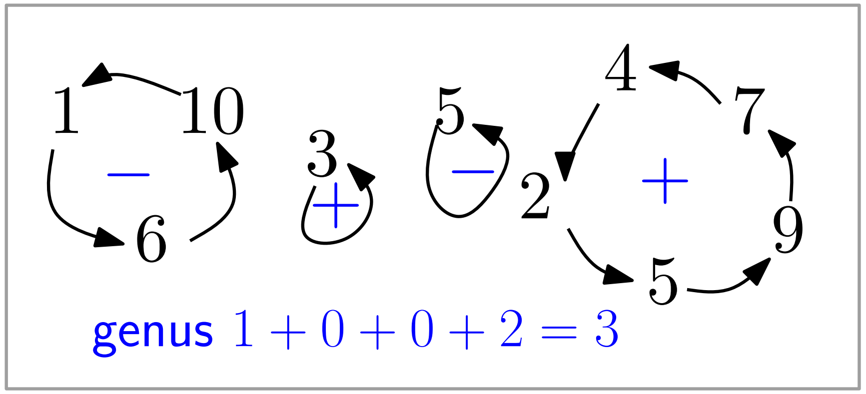


- **Theorem** [C., Féray, Fusy] (our main result!)

There is a 2^{n+1}-to-1-jection between unicellular maps of size n and C-decorated trees with n edges. It preserves both the genus and the underlying graph.

Our result, in short

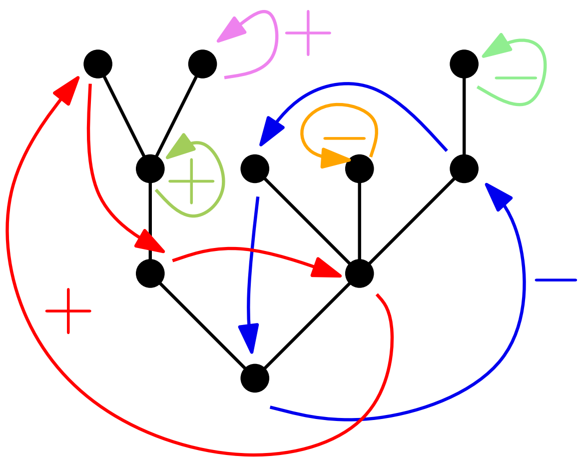
- A C-permutation of a set S:
  - all cycles have odd length
  - each cycle carries a sign in $\{+,-\}$
  - its genus is $g := \sum_i k_i$ where $(2k_i+1)$ are the cycle-lengths

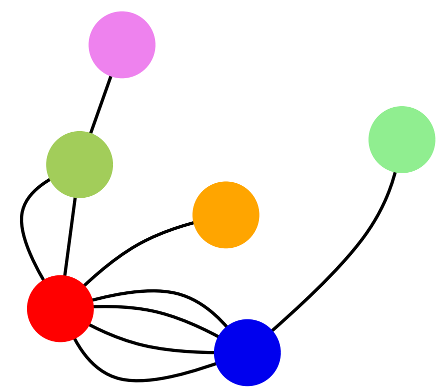


$$\#\text{cycles} = |S| - 2g$$

- C-decorated tree = a rooted plane tree equipped with a C-permutation of its vertices:

a C-decorated tree of genus 2





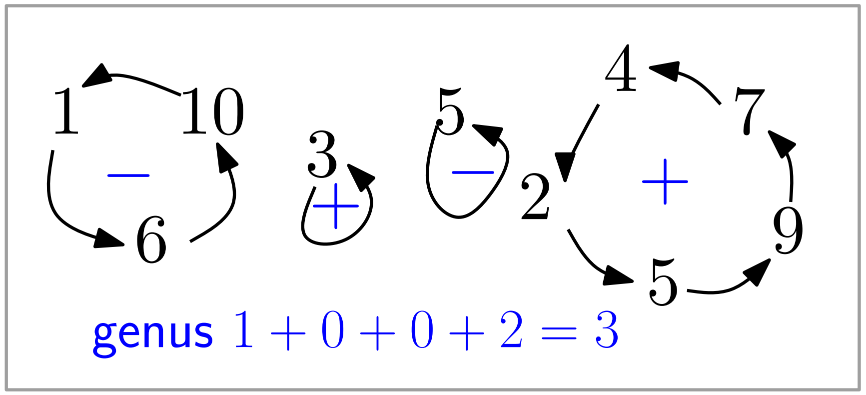
the underlying graph is obtained by identifying vertices in each cycle into a big vertex

- **Theorem** [C., Féray, Fusy] (our main result!)

There is a 2^{n+1}-to-1-jection between unicellular maps of size n and C-decorated trees with n edges. It preserves both the genus and the underlying graph.

Our result, in short

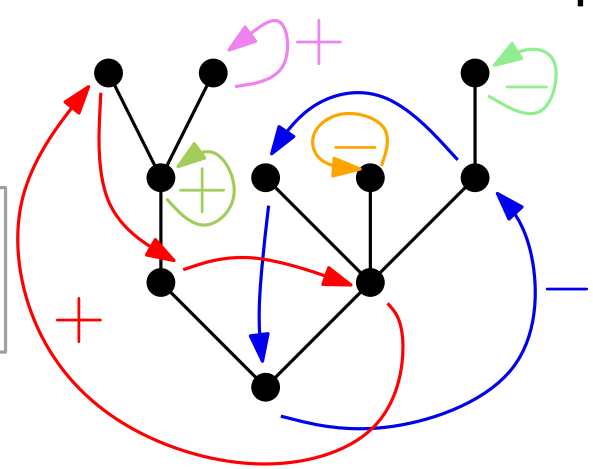
- A C-permutation of a set S:
  - all cycles have odd length
  - each cycle carries a sign in $\{+,-\}$
  - its genus is $g := \sum_i k_i$ where $(2k_i+1)$ are the cycle-lengths

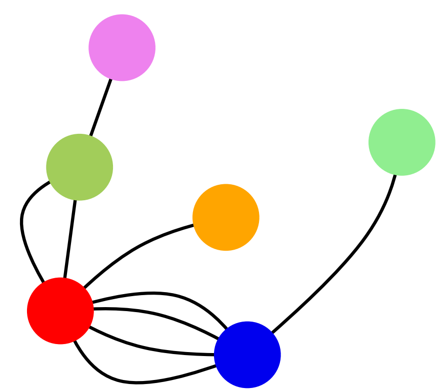


$$\#\text{cycles} = |S| - 2g$$

- C-decorated tree = a rooted plane tree equipped with a C-permutation of its vertices:

a C-decorated tree of genus 2





the underlying graph is obtained by identifying vertices in each cycle into a big vertex

- **Theorem** [C., Féray, Fusy] (our main result!)

There is a 2^{n+1}-to-1-jection between unicellular maps of size n and C-decorated trees with n edges. It preserves both the genus and the underlying graph.

FROM THERE ALL KNOWN FORMULAS FOLLOW ON, BIJECTIVELY!

A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

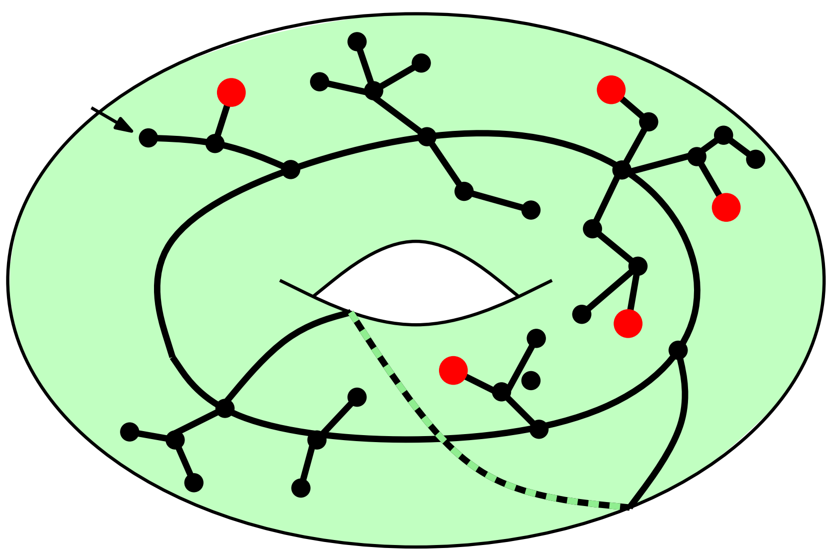
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



genus 1, 5 marked vertices

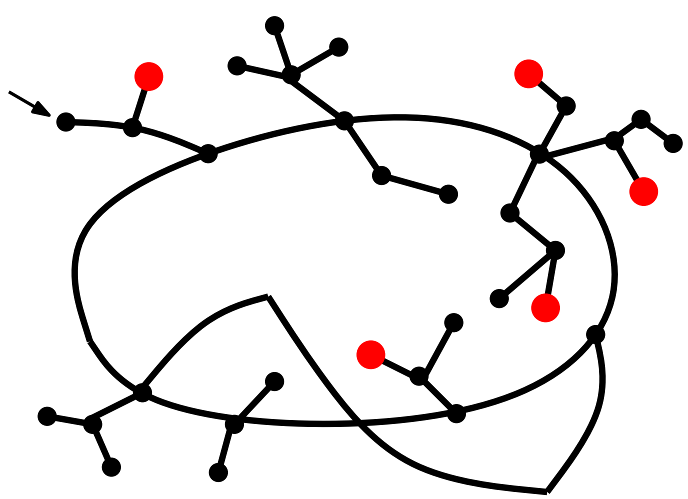
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



genus 1, 5 marked vertices

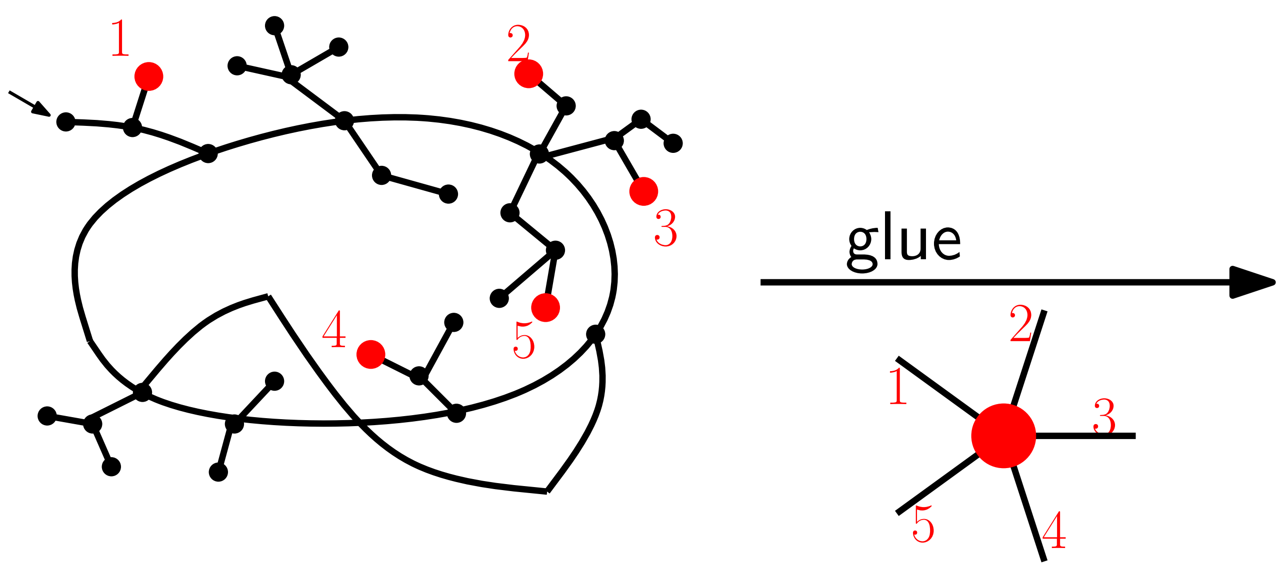
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



genus 1, 5 marked vertices

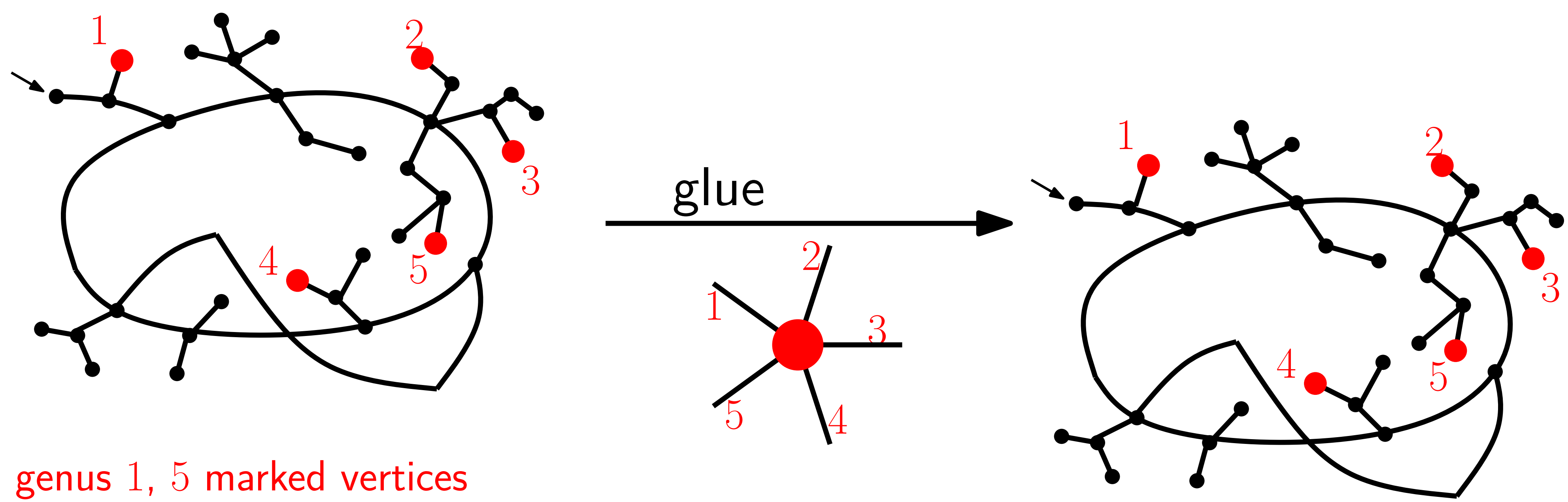
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



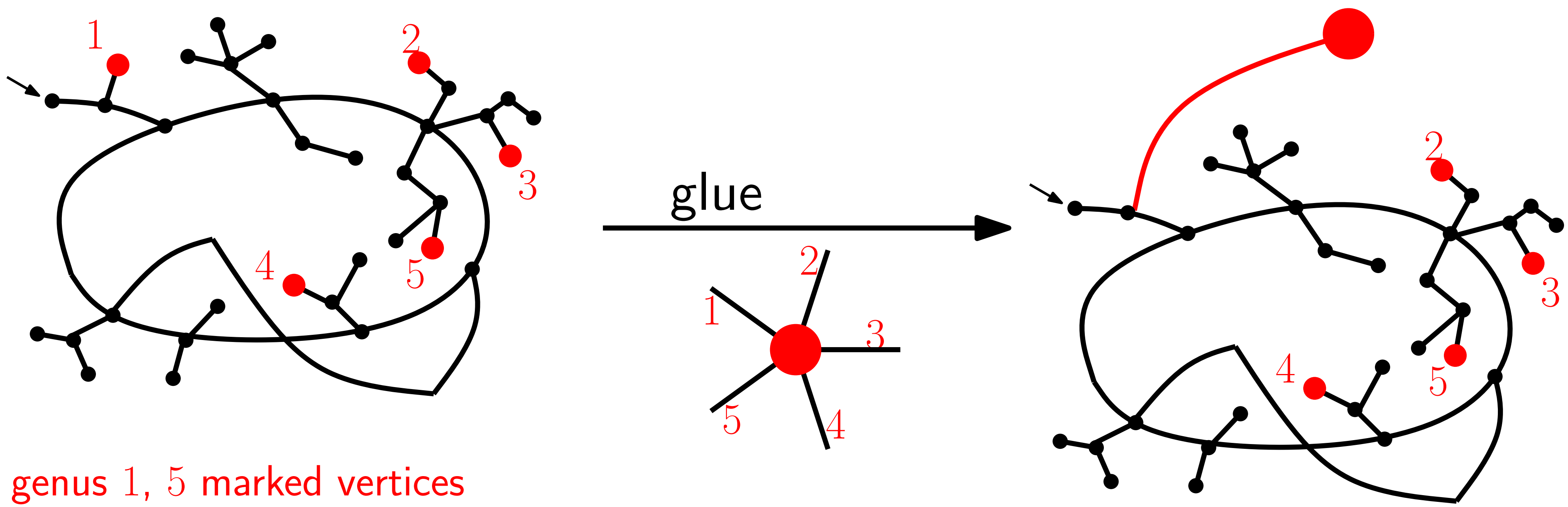
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



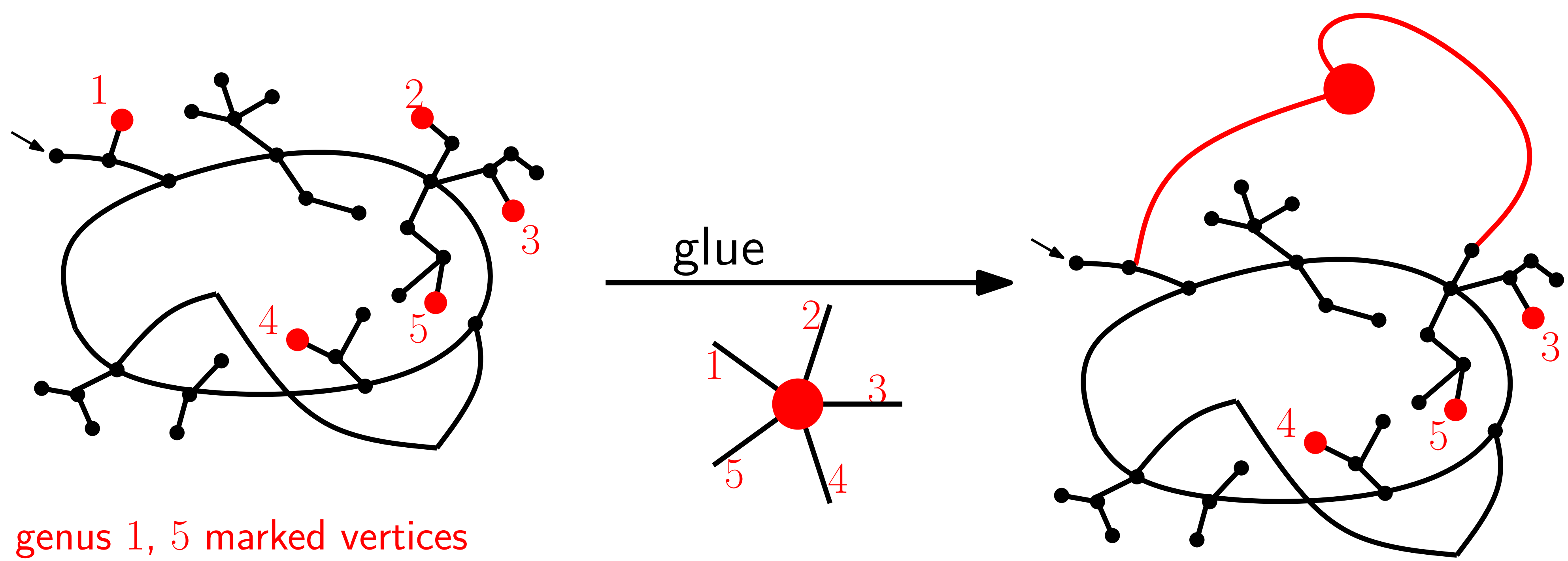
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



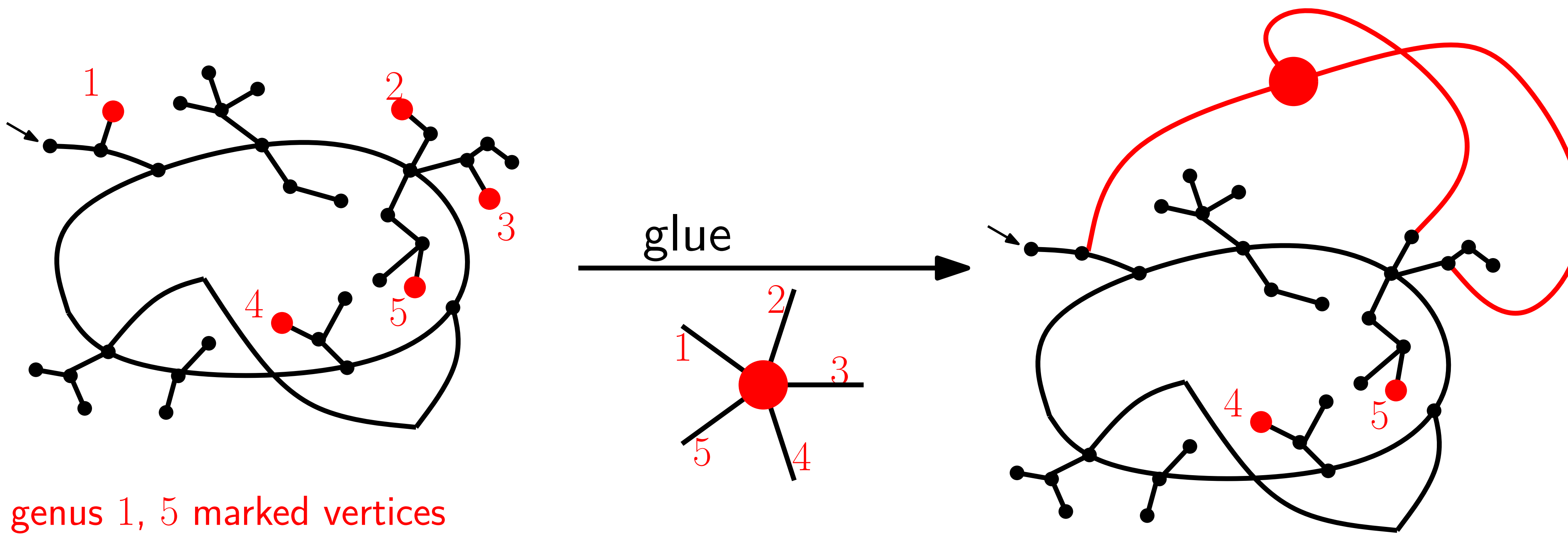
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



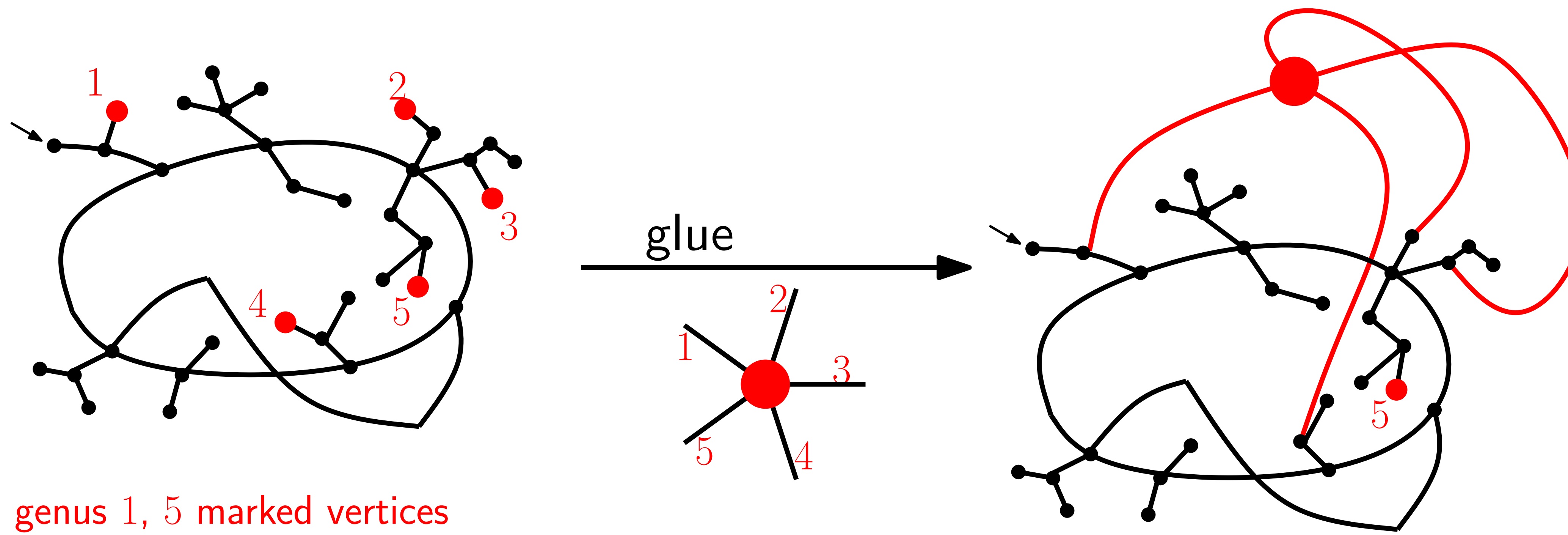
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



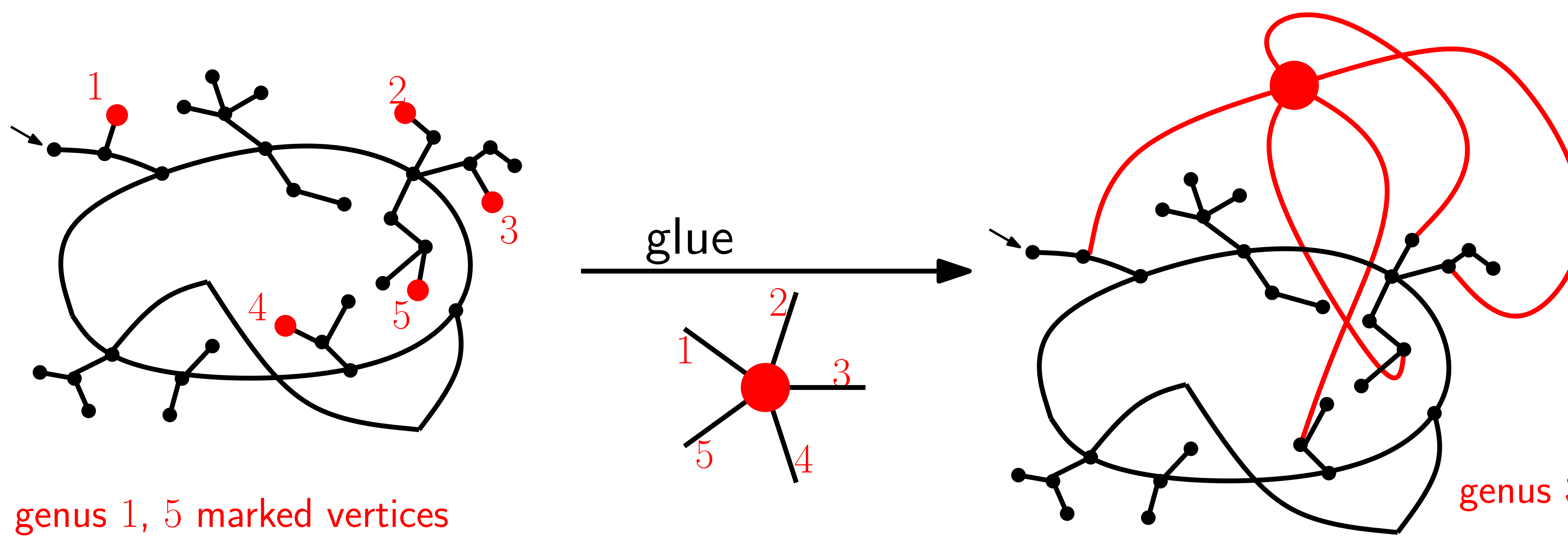
A bijection from FPSAC'09 (1)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Sketch:** Take $(2k+1)$ vertices in a map of genus $g-k$ and glue them together preserving the "one-face" condition:



- the genus increases by k (Euler's formula)

A bijection from FPSAC'09 (2)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

A bijection from FPSAC'09 (2)

Let $\mathcal{E}_g^{(k)}(n)$ = unicellular maps, genus g, n edges, k marked vertices.

- **Theorem** [Ch.09] There is an explicit $2g$-to-1-jection that realizes:

$$2g \cdot \mathcal{E}_g(n) = \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n)$$

- **Corollary:** $\epsilon_g(n) = P_g(n) \times \mathrm{Cat}(n)$ where the polynomial P_g is defined recursively:

$$2g \cdot P_g(n) = \binom{n+3-2g}{3} P_{g-1}(n) + \binom{n+5-2g}{5} P_{g-2}(n) + \cdots + \binom{n+1}{2g+1} P_0(n)$$

...but now we can say more !

C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n) = C$-permutations, size n, genus g, k marked cycles.

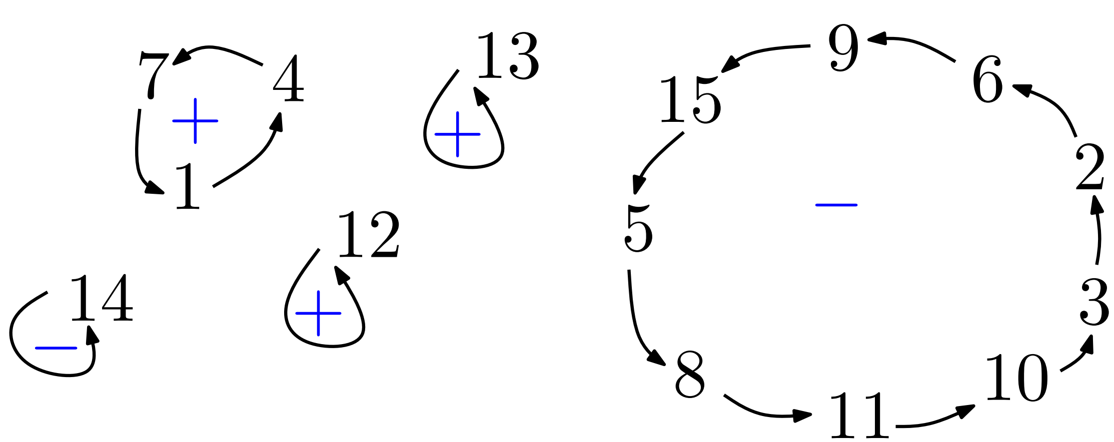
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n) = C$-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



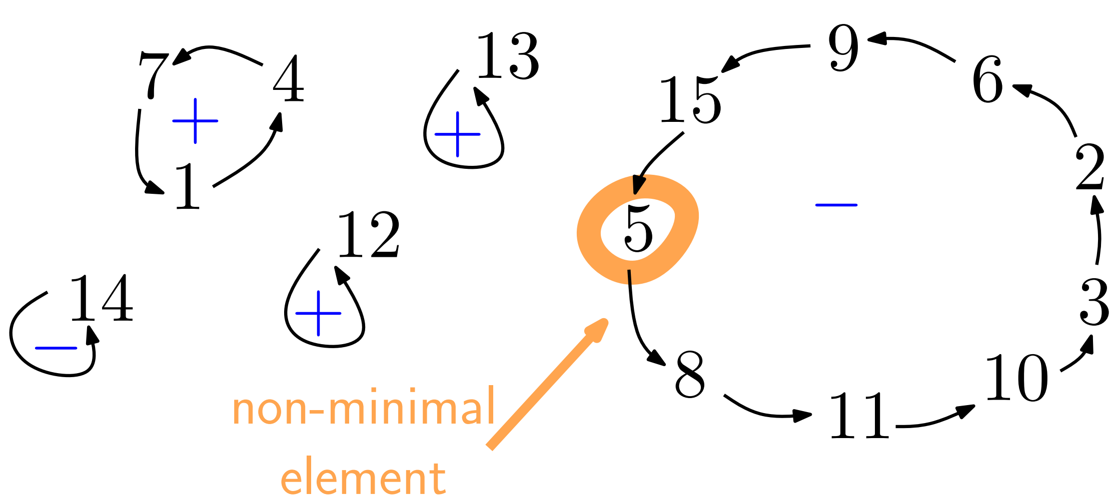
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n)$ = C-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
$2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



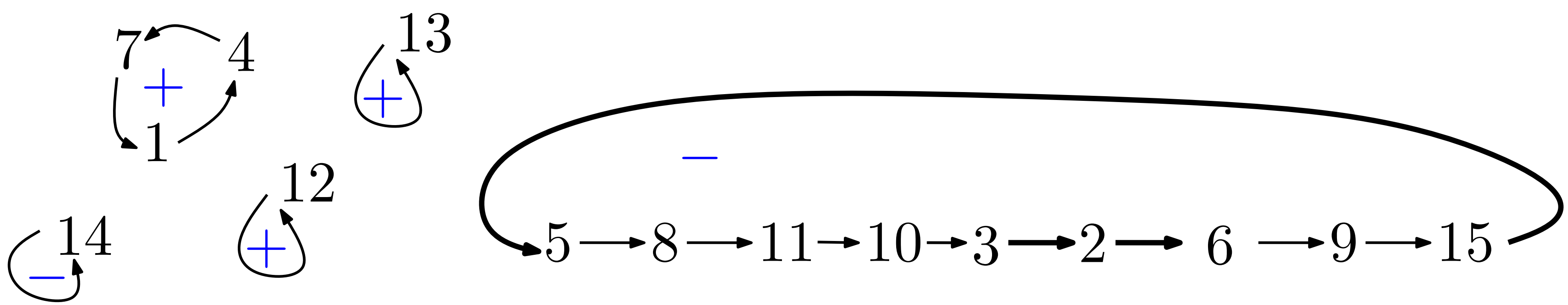
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n) = C$-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



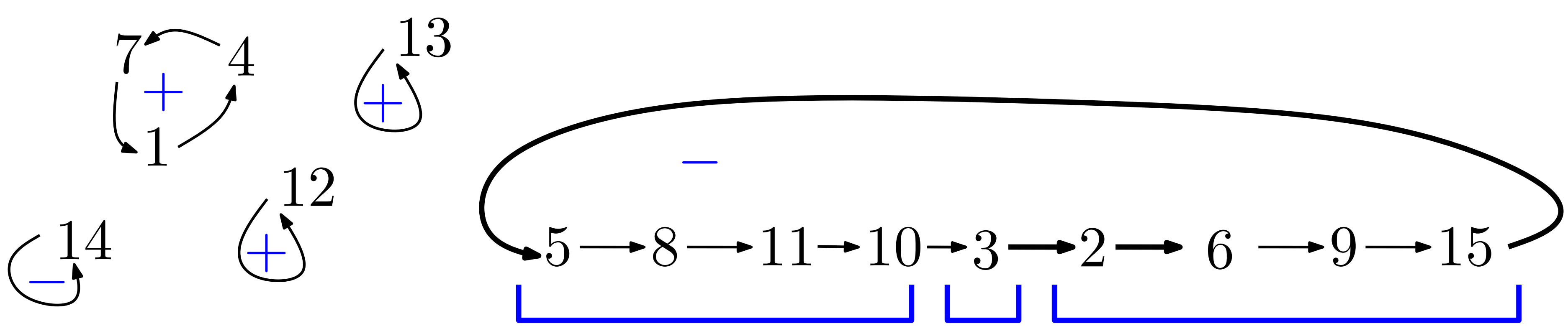
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}^{(3)}_{g-1}(n) + \mathcal{C}^{(5)}_{g-2}(n) + \cdots + \mathcal{C}^{(2g+1)}_0(n)$$

where $\mathcal{C}^{(k)}_g(n) =$ C-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...

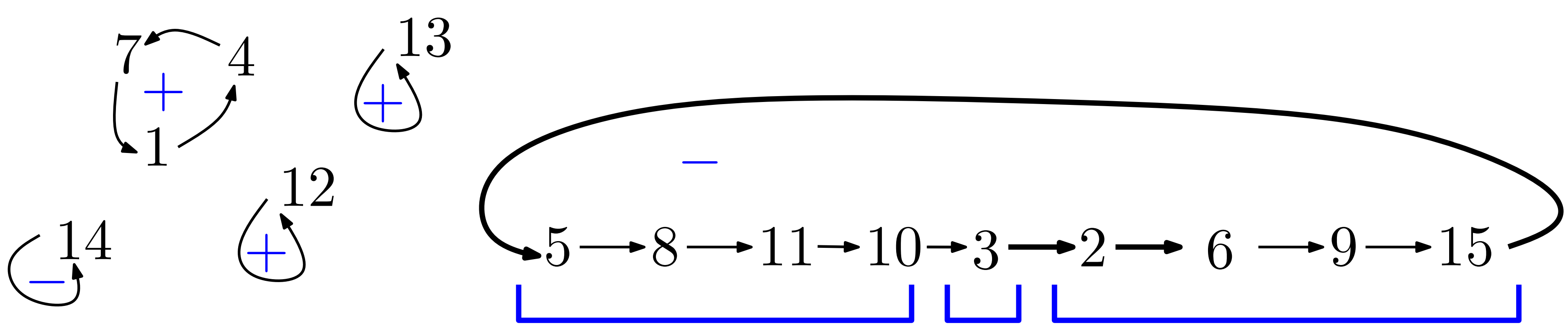
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n) = C$-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...
   correct this by applying a small transposition when necessary
   and write a "+" to remember that you have done it

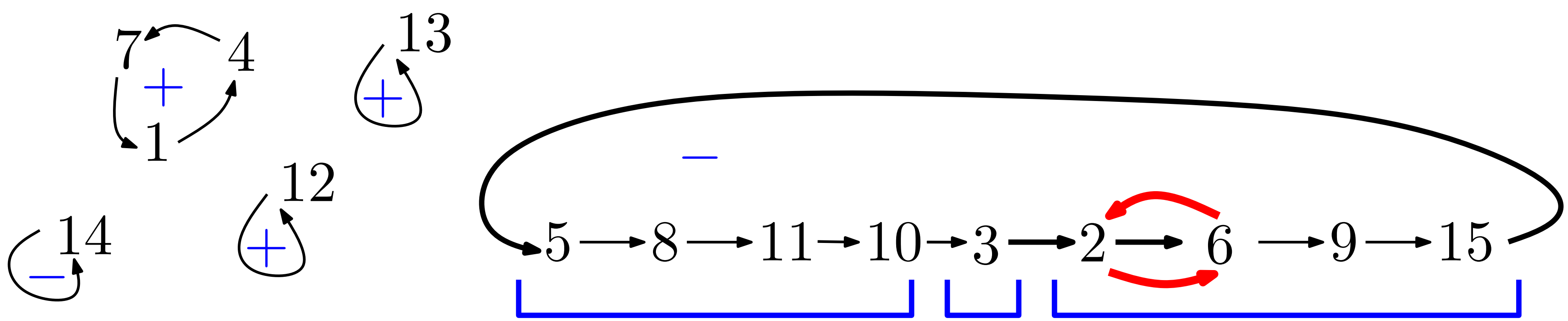
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n)$ = C-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...
   correct this by applying a small transposition when necessary
   and write a "+" to remember that you have done it

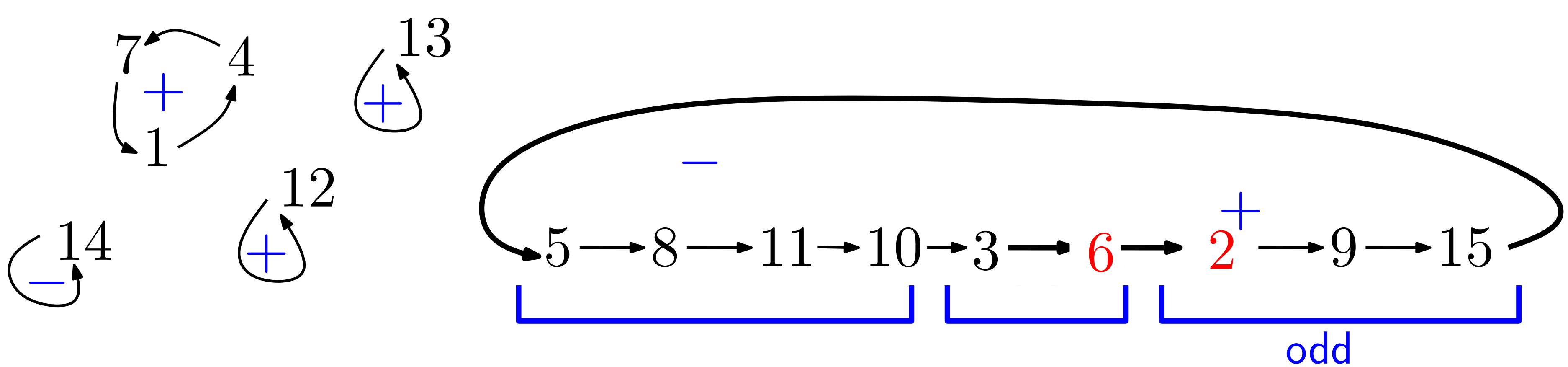
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n) = C$-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...
   correct this by applying a small transposition when necessary
   and write a "$+$" to remember that you have done it

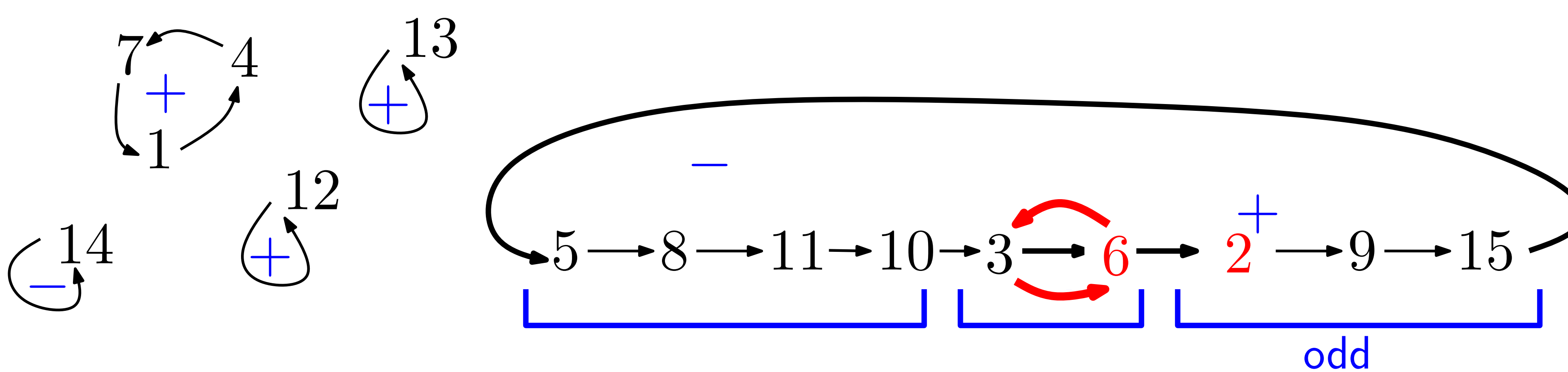
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n) = C$-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...
   correct this by applying a small transposition when necessary
   and write a "$+$" to remember that you have done it

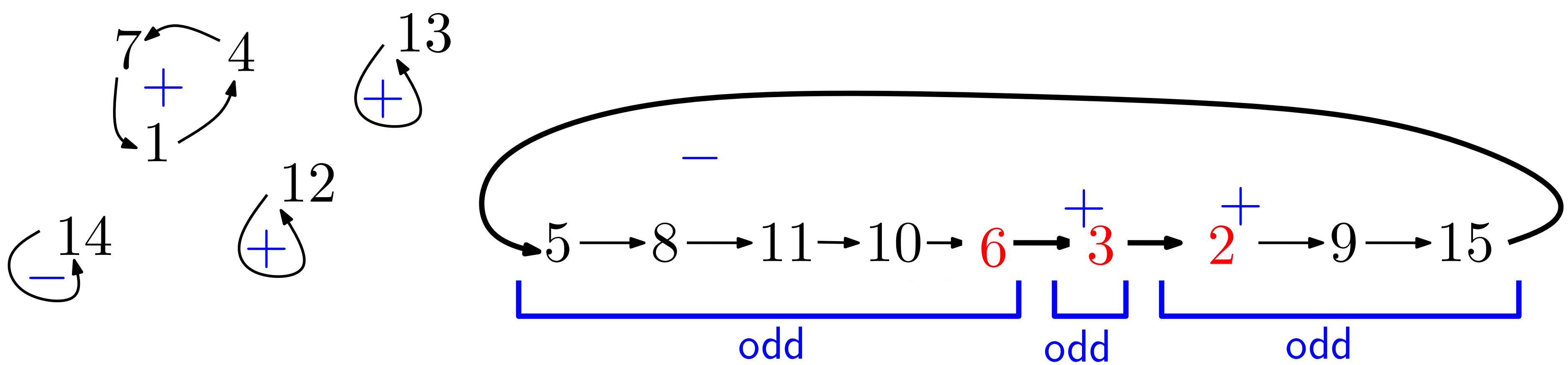
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n)$ = C-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...
   correct this by applying a small transposition when necessary
   and write a "+" to remember that you have done it

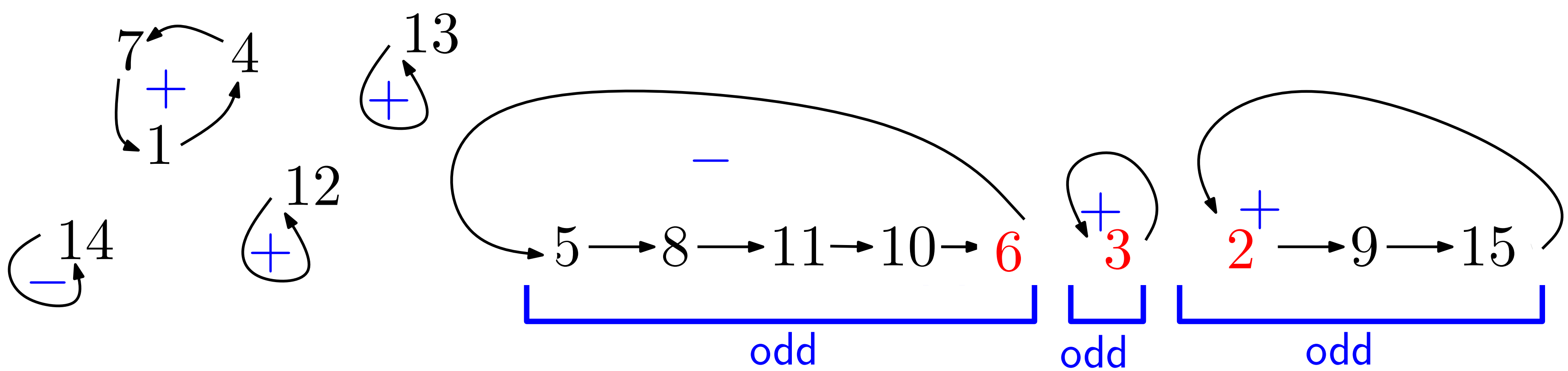
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n)$ = C-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...
   correct this by applying a small transposition when necessary
   and write a "+" to remember that you have done it
3. now close each run into a cycle.

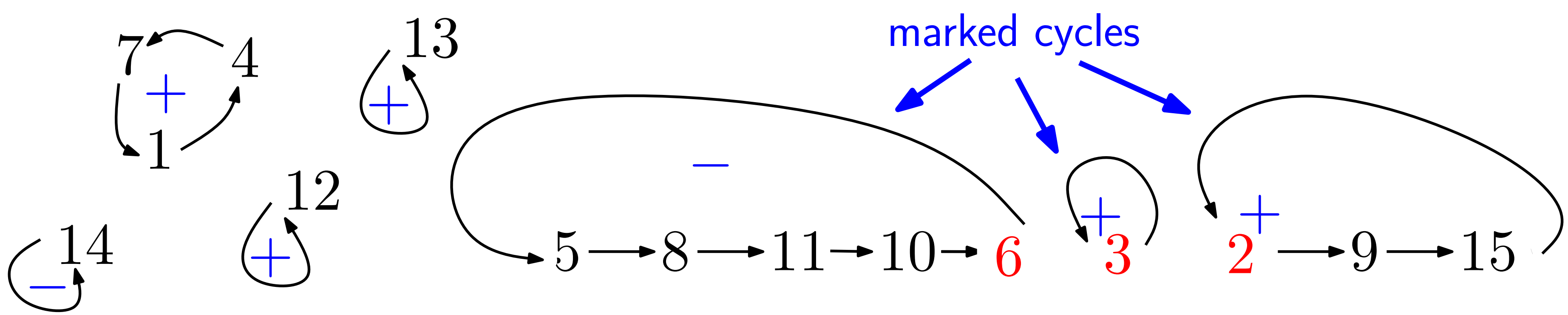
C-permutations solve the recurrence combinatorially!

- **Fact:** C-permutations satisfy the same recurrence as unicellular maps:

$$2g \cdot \mathcal{C}_g(n) = \mathcal{C}_{g-1}^{(3)}(n) + \mathcal{C}_{g-2}^{(5)}(n) + \cdots + \mathcal{C}_0^{(2g+1)}(n)$$

where $\mathcal{C}_g^{(k)}(n)$ = C-permutations, size n, genus g, k marked cycles.

- **Sketch (left-to-right):** Start with a C-permutation of genus g.
  $2g = (\#\text{elements}) - \#(\text{cycles})$ is the number of non-minimal elements



1. look at runs separating lower records
2. some may have even length...
   correct this by applying a small transposition when necessary
   and write a "+" to remember that you have done it
3. now close each run into a cycle.

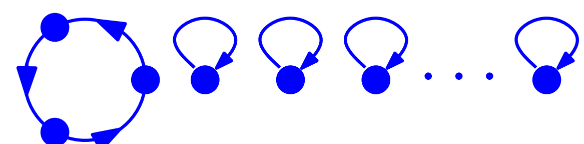
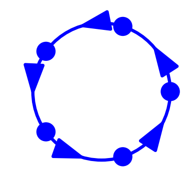
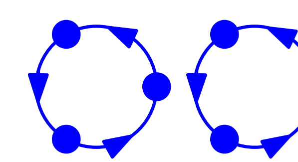
Counting C-decorated trees is straightforward

- **Theorem** [C., Féray, Fusy]

The number of unicellular maps of genus g with n edges satisfies:

$$2^{n+1}\epsilon_g(n) = C_g(n+1)\text{Cat}(n)$$

where $C_g(n+1)$ is the number of C-perm. of genus g on $n+1$ elements.

- but $C_g(n+1) =$ easy numbers!
  - $C_0(n+1) = 2^{n+1}$  ($n+1$ cycles)
  - $C_1(n+1) = \frac{(n+1)n(n-1)}{3} 2^{n-1}$  ($n-1$ cycles)
  - $C_2(n+1) = \left(4!\binom{n+1}{5} + 40\binom{n+1}{6}\right)2^{n-3}$ (either  or )

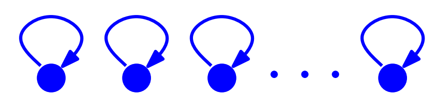
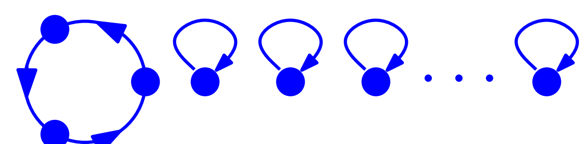
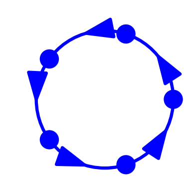
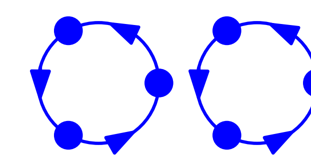
Counting C-decorated trees is straightforward

- **Theorem** [C., Féray, Fusy]

The number of unicellular maps of genus g with n edges satisfies:

$$2^{n+1}\epsilon_g(n) = C_g(n+1)\mathrm{Cat}(n)$$

where $C_g(n+1)$ is the number of C-perm. of genus g on $n+1$ elements.

- but $C_g(n+1)$ = easy numbers!
  - $C_0(n+1) = 2^{n+1}$  ($n+1$ cycles)
  - $C_1(n+1) = \frac{(n+1)n(n-1)}{3}2^{n-1}$  ($n-1$ cycles)
  - $C_2(n+1) = \left(4!\binom{n+1}{5} + 40\binom{n+1}{6}\right)2^{n-3}$ (either  or )

- In general: $C_g(n+1) = \left(\sum_{\gamma\vdash g} \frac{(n+1-2g)_{2\ell(\gamma)+1}}{\prod_i m_i!(2i+1)^{m_i}}\right)2^{n+1-2g}$

sum is over the cyle type of the C-permutation: $(2\gamma_i+1)$ = cycle lengths.

$\surd$ Lehman-Walsh formula !

Conclusion

- It is a series of exercises to recover ALL the known formulas concerning unicellular maps, bijectively with C-decorated trees. You just need to know your classics (count permutations, count trees...). Take a look at the paper!

- For example the beautiful Harer-Zagier recurrence formula has been waiting for a combinatorial interpretation since 1986...

Harer-Zagier recurrence formula (1986)

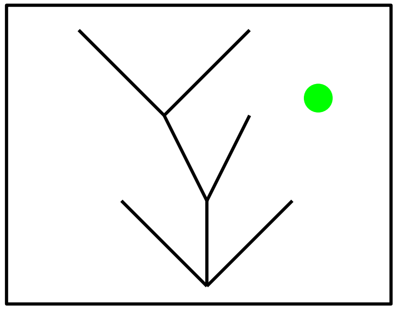
- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$

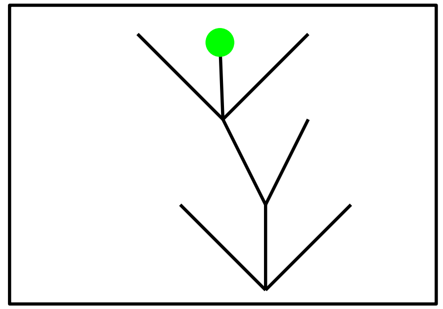
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

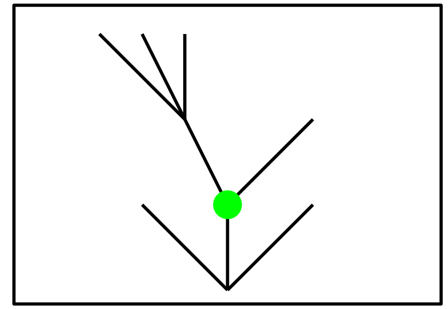
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



case a: vertex is a leaf. Delete it.

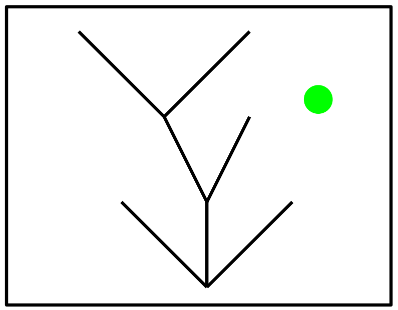


case b: vertex is not a leaf. Contract the leftmost outgoing edge.

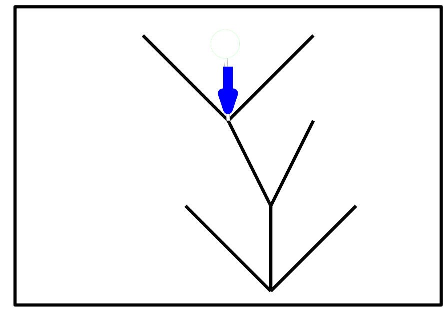
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

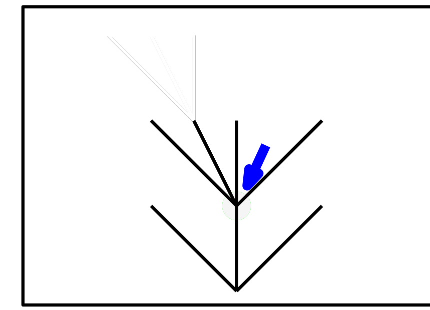
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



case a: vertex is a leaf. Delete it.

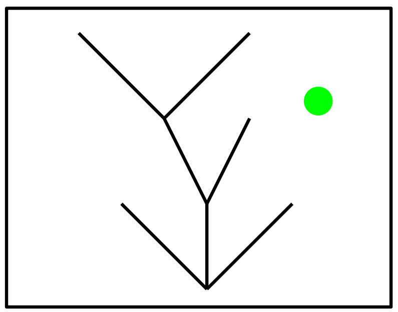


case b: vertex is not a leaf. Contract the leftmost outgoing edge.

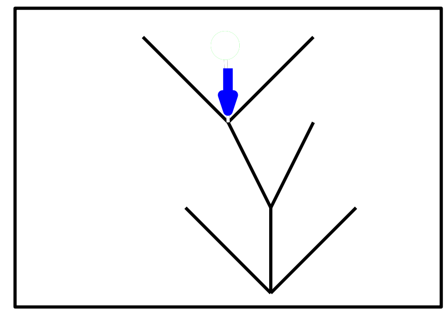
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

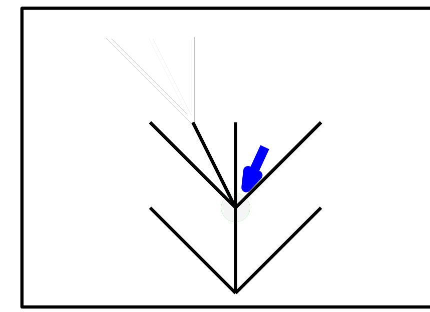
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



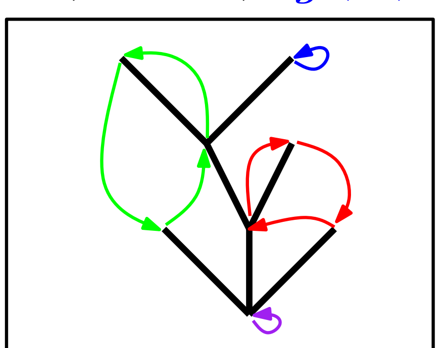
case a: vertex is a leaf. Delete it.



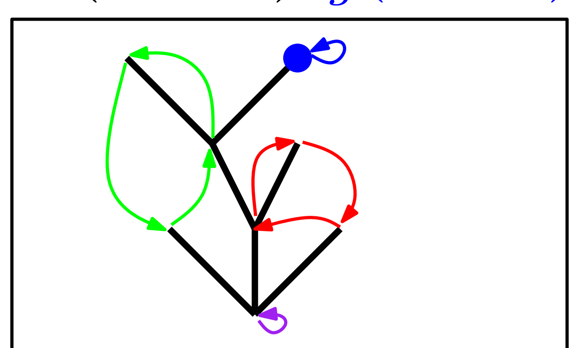
case b: vertex is not a leaf. Contract the leftmost outgoing edge.

- Then for general g:

$$(n+1)\epsilon_g(n) \quad = \quad 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$



C-decorated tree, n edges, genus g, one marked vertex



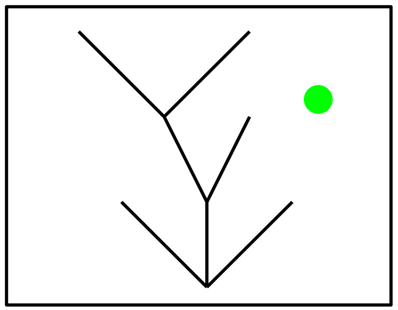
case 1: vertex is a fixed point: apply Rémy's bijection (one vertex disappears)

case 2: vertex is in a $(2k+1)-$cycle. Apply Rémy's bijection twice (two vertices disappear, cycle length decreases by 2)

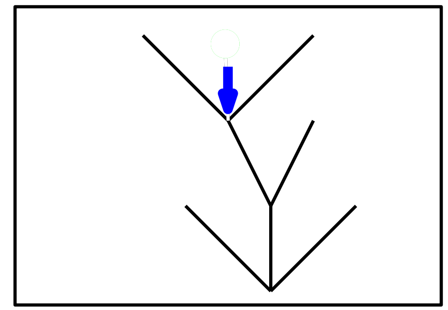
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

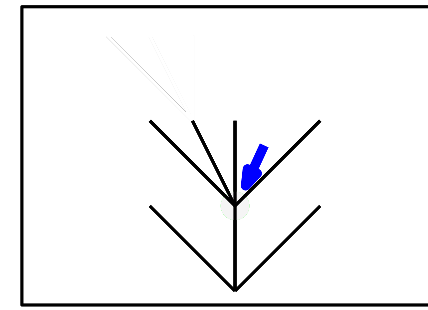
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



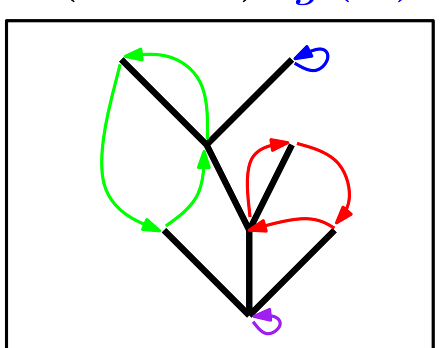
case a: vertex is a leaf. Delete it.



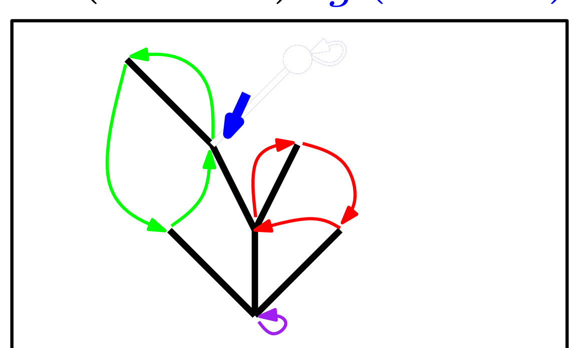
case b: vertex is not a leaf. Contract the leftmost outgoing edge.

- Then for general g:

$$(n+1)\epsilon_g(n) \quad = \quad 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$



C-decorated tree, n edges, genus g, one marked vertex



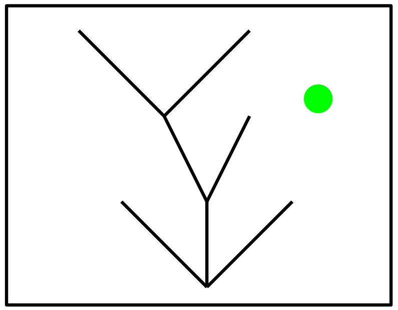
case 1: vertex is a fixed point: apply Rémy's bijection (one vertex disappears)

case 2: vertex is in a $(2k+1)-$cycle. Apply Rémy's bijection twice (two vertices disappear, cycle length decreases by 2)

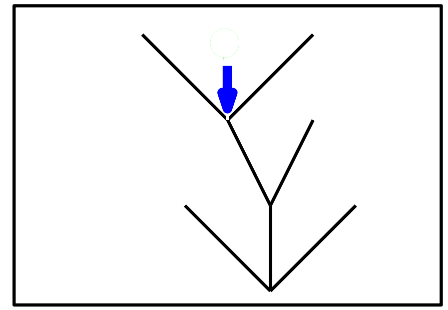
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

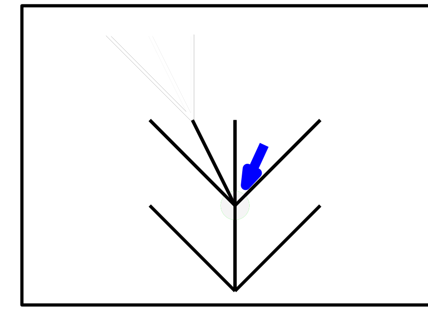
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



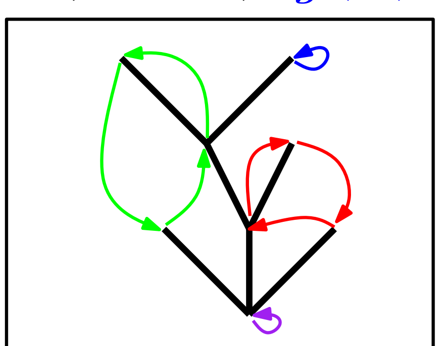
case a: vertex is a leaf. Delete it.



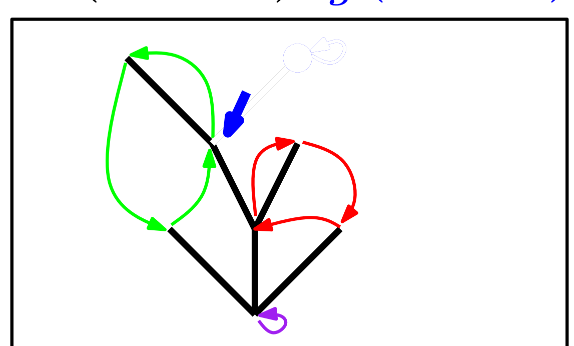
case b: vertex is not a leaf. Contract the leftmost outgoing edge.

- Then for general g:

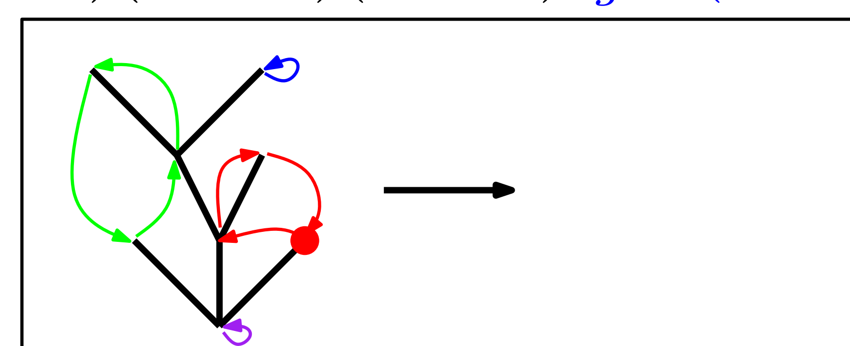
$$(n+1)\epsilon_g(n) \quad = \quad 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$



C-decorated tree, n edges, genus g, one marked vertex



case 1: vertex is a fixed point: apply Rémy's bijection (one vertex disappears)

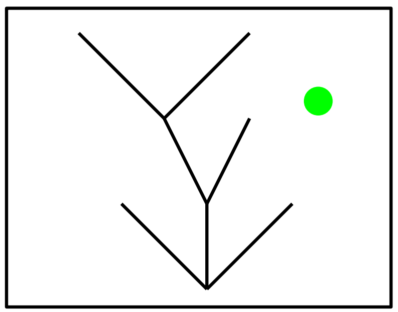


case 2: vertex is in a $(2k+1)$−cycle. Apply Rémy's bijection twice (two vertices disappear, cycle length decreases by 2)

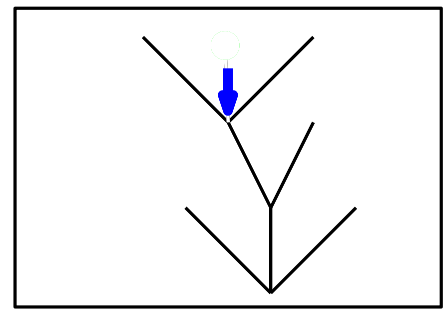
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

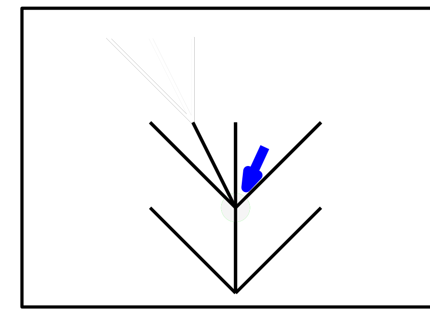
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



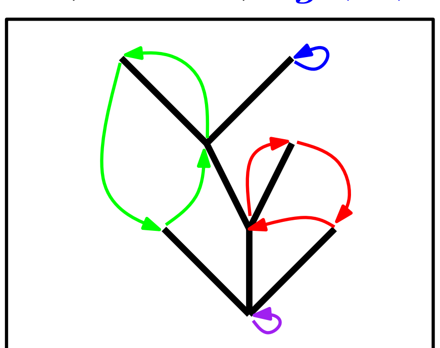
case a: vertex is a leaf. Delete it.



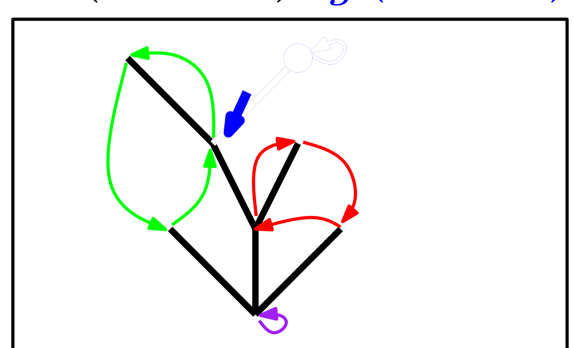
case b: vertex is not a leaf. Contract the leftmost outgoing edge.

- Then for general g:

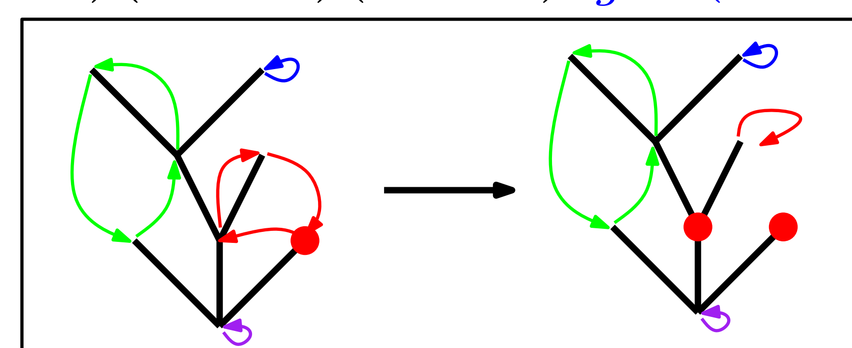
$$(n+1)\epsilon_g(n) \quad = \quad 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$



C-decorated tree, n edges, genus g, one marked vertex



case 1: vertex is a fixed point: apply Rémy's bijection (one vertex disappears)

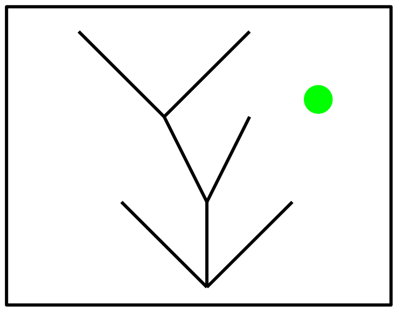


case 2: vertex is in a $(2k+1)$−cycle. Apply Rémy's bijection twice (two vertices disappear, cycle length decreases by 2)

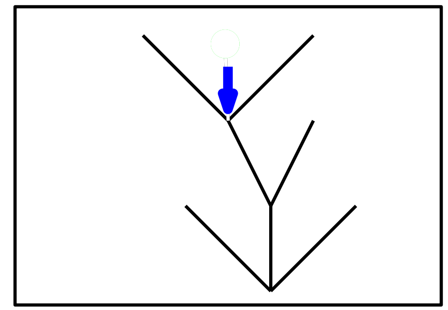
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

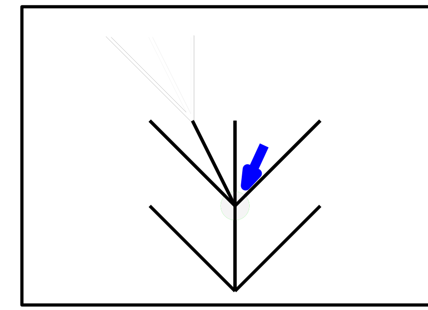
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



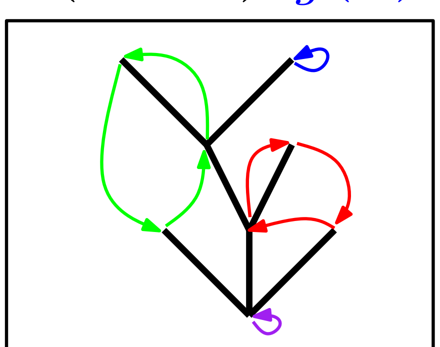
case a: vertex is a leaf. Delete it.



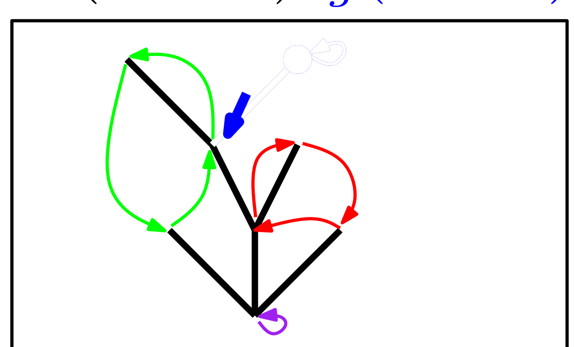
case b: vertex is not a leaf. Contract the leftmost outgoing edge.

- Then for general g:

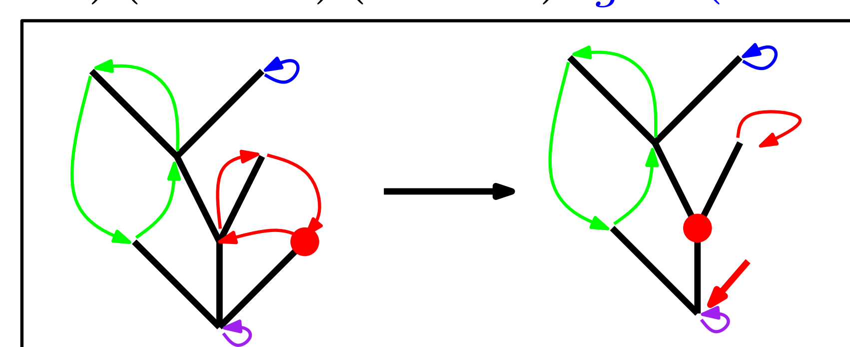
$$(n+1)\epsilon_g(n) \quad = \quad 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$



C-decorated tree, n edges, genus g, one marked vertex



case 1: vertex is a fixed point: apply Rémy's bijection (one vertex disappears)

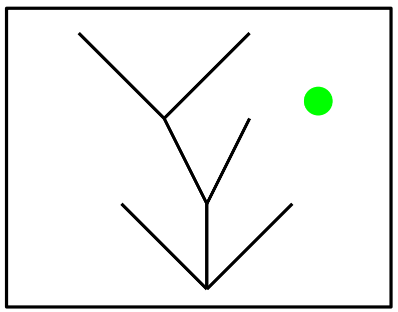


case 2: vertex is in a $(2k+1)-$cycle. Apply Rémy's bijection twice (two vertices disappear, cycle length decreases by 2)

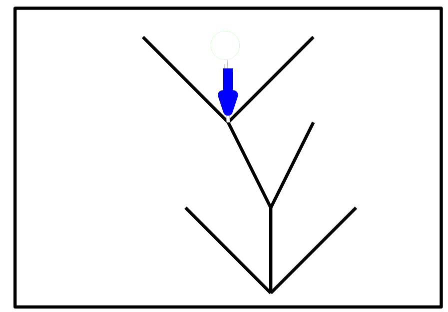
Harer-Zagier recurrence formula (1986)

- Classic: for $g = 0$, Rémy's bijection [Rémy 85]

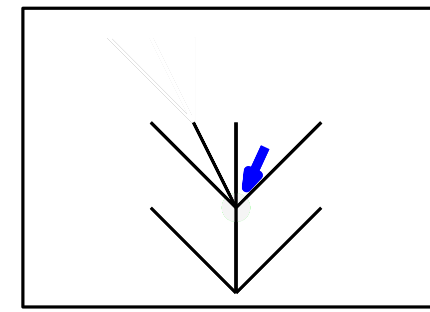
$$(n+1)\mathrm{Cat}(n) \quad = \quad 2 \times (2n-1)\mathrm{Cat}(n-1)$$



rooted tree, n edges, one marked vertex



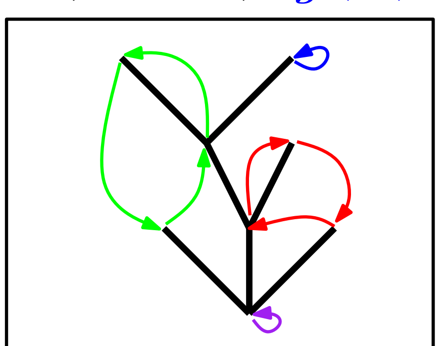
case a: vertex is a leaf. Delete it.



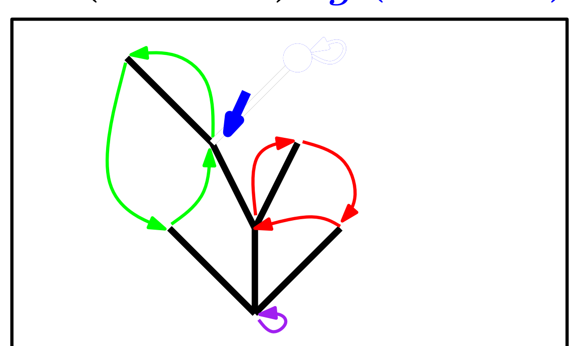
case b: vertex is not a leaf. Contract the leftmost outgoing edge.

- Then for general g:

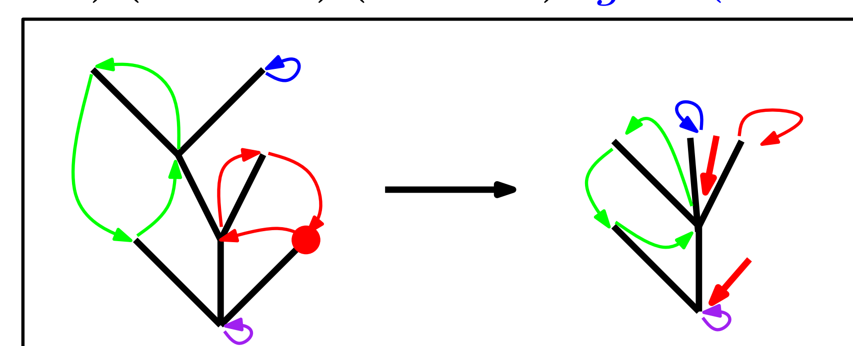
$$(n+1)\epsilon_g(n) \quad = \quad 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2)$$



C-decorated tree, n edges, genus g, one marked vertex



case 1: vertex is a fixed point: apply Rémy's bijection (one vertex disappears)



case 2: vertex is in a $(2k+1)-$cycle. Apply Rémy's bijection twice (two vertices disappear, cycle length decreases by 2)

Conclusion

• It is a series of exercises to recover ALL the known formulas concerning unicellular maps, bijectively with C-decorated trees. You just need to know your classics (count permutations, count trees...). Take a look at the paper!

• The bijection also applies to Féray's expression of characters in terms of unicellular maps (we obtain a new expression - is it useful?)

• Next ?

$\rightarrow$ unicellular constellations? ([Poulalhon-Schaeffer 02, Bernardi-Morales 11])
(problem: FPSAC'09 bijection does not work well)
(very partial results in the full version - take a look!)

$\rightarrow$ many-face maps? (KP hierarchy?)
(problem: seems much harder!)

$\rightarrow$ non-orientable surfaces?
(problem: FPSAC'09 bijection only exists in asymptotic version - [Bernardi-Ch., FPSAC'10])