On some tree-structured graphs
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Schedule

- Many combinatorial problems can be solved on trees (not all) using their simple structure.
- Either using a bottom-up or a top-down approach. Divide and conquer approach.
- Exercise : find some NP-hard problems on trees.
- This explain the study of "tree-structured" graphs.

Some examples in this course

1. Cographs (see first course)
2. Modular decomposition and split decomposition
3. Chordal graphs
4. Treewidth and other width graph parameters such as cliquewidth, branchwitdth and rankwidth
5. Treelength
6. δ-hyperbolicity, distance to a tree in a metric way (by Gromov)
7. ...

Modules

Modules

For a graph $G = (V, E)$, a **module** is a subet of vertices $A \subseteq V$ such that
$\forall x, y \in A,\ N(x) - A = N(y) - A$
The problem with this definition : must we check all subsets A ?

Trivial Modules

$\emptyset$, $\{x\}$ and V are modules.

Prime Graphs

A graph is **prime** if it admits only trivial modules.

Application of divide and conquer

Let M be a non trivial module of a graph and $m \in M$. We define G_m the graph obtain from G by contracting M into a single vertex m. If we want to compute $\omega(G)$, the maximal clique size in G

- One can compute first $\omega(G(M))$ where $G(M)$ is the subgraph of G induced by M
- And then compute $\omega_w(G_m)$, where $\omega_w(G)$ represents the maximum weight of a clique in G, using the weight function w.
- We choose w as follows : $\forall x \in V(G_m)$, if $x \neq m$ then $w(x) = 1$ and $w(m) = \omega(G(M))$.
- So we have reduce the original problem into 2 instances of the problem on smaller graphs.

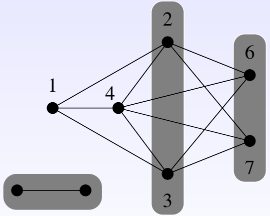
- Our main goal is to find good algorithms for modular decomposition.
  But we cannot avoid to investigate in details the combinatorial properties of the modules in graphs.
- Of course modules can be also defined for directed graphs but also for many discretes structures such as hypergraphs, matroids, boolean functions, submodular functions, automaton, . . .

Examples

Characterization of Modules

A subset of vertices M of a graph $G = (V, E)$ is a **module** iff

$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$



Examples of modules

- connected components of G
- connected components of $\overline{G}$
- any vertex subset of the complete graph (or the stable)

- Modular decomposition (algorithmic aspects)
- But also an operation on graphs : Modular composition
  a graph grammar with a simple rule : replace a vertex by a graph
- Very natural notion, (re)discovered under many names in various combinatorial structures
  such as : clan, homogeneous set, ...
- An important tool in graph theory

Playing with the definition

Duality

A is a module of G implies A is a module of $\overline{G}$.

Easy observations

- No prime undirected graph with ≤ 3 vertices (false for directed graphs, as a directed triangle shows it)
- P_4 the path with 4 vertices is the only prime on 4 vertices.
- P_4 is isomorphic to its complement.

Twins and strong modules

Twins

$x, y \in V$ are false- (resp. true-) **twins** if $N(x) = N(y)$ (resp. $N(x) \cup \{x\} = N(y) \cup \{y\}$.
x, y are false twins in G iff x, y are true twins in $\overline{G}$.
Classes of twins are particular modules (stable sets for false twins and complete for true twins).

Strong modules

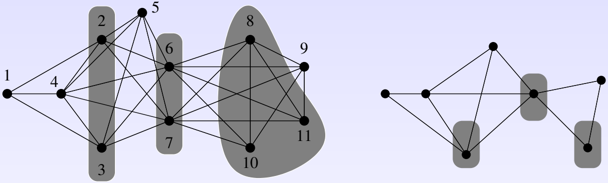
A **strong module** is a module that does not strictly overlap any other module.

2 important facts about twins

1. They allow to define the class of cographs
2. False twins are the only modules of bipartite graphs. Some approximation can be done with many applications in community detection on bipartite graphs.

Modular partition

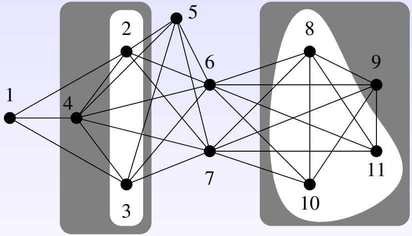
A partition $\mathcal{P}$ of the vertex set of a graph $G = (V, E)$ is a **modular partition** of G if any part is a module of G.

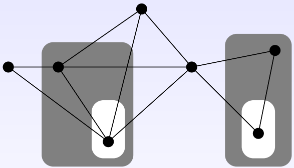


Let $\mathcal{P}$ be a modular partition of a graph $G = (V, E)$. The **quotient graph** $G_{/\mathcal{P}}$ is the induced subgraph obtained by choosing one vertex per part of $\mathcal{P}$.

Lemma (Mohring Radermacher 1984)

Let $\mathcal{P}$ be a modular partition of $G = (V, E)$.
$\mathcal{X} \subseteq \mathcal{P}$ is a module of $G_{/\mathcal{P}}$ iff $\cup_{M \in \mathcal{X}} M$ is a module of G.





Modular Decomposition Theorem

Theorem (Gallai 1967)

Let $G = (V, E)$ be a graph with $|V| \geq 4$, the three following cases are mutually exclusive :

1. *G is not connected,*
2. *$\overline{G}$ is not connected,*
3. *$G_{/\mathcal{M}(G)}$ is a prime graph, with $\mathcal{M}(G)$ the modular partition containing the maximal strong modules of G.*

As a byproduct, we notice that a prime graph G satisfies :
G and $\overline{G}$ are connected

Modular decomposition tree

Tree

A recursive a this theorem yields a tree T in which :

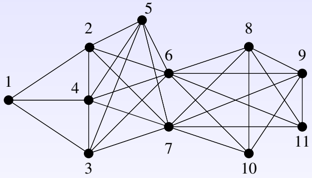
- The root corresponds to V
- Leaves are associated to vertices
- Each node corresponds to a strong module

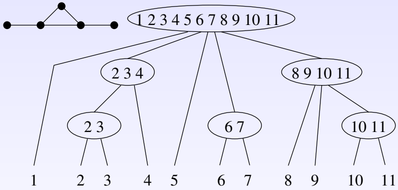
There are 3 types of nodes :

Parallel, Series and Prime

Another explanation

The set of strong modules is nested into an inclusion tree (called the **modular decomposition tree** $MD(G)$ of G).





Partitive Families

Lemma

If M and M' are two overlapping modules then

- *(i) $M \setminus M'$ is a module*
- *(ii) $M \cap M'$ is a module*
- *(iii) $M \cup M'$ is a module*
- *(iv) $M \Delta M'$ is a module*

- A family satisfying (i) - (iv) is called a partitive family
- A family satisfying (i) - (iii)) is called a weak partitive family

Remarks on the module properties

- (ii) is always true, even if M and M' are not overlapping
- (iii) could be written that way :
  If $M \cap M' \neq \emptyset$ then $M \cup M'$ is a module
- (iv) is not a consequence of (i) and (iii) since $M \setminus M'$ and $M' \setminus M$ do not overlap. And overlapping is really needed to prove (i) for modules.

A proof of Gallai's theorem

We note that the two first cases are exclusive. Either G is not connected or $\overline{G}$ is not connected but not both. Let us consider the third case in which G and $\overline{G}$ are connected.

1. If G is prime, then the third case is trivially obtained.
2. Else G admits some non trivial modules. Let us consider $M_1, \ldots M_k$ the maximal non trivial modules of G.
   We will prove that they are strong modules.

- if M_i and M_j overlap then using the algebraic properties of modules : necessarily $M_i \cup M_j$ is a module.
- But using their maximality : $M_i \cup M_j = V(G)$.
- Then $(M_i \setminus M_j, M_i \cap M_j, M_j \setminus M_i)$ is a modular partition of G.

- Since G is connected, there is at least one edge between these 3 sets. But then using the definition of modules all edges between these 3 sets exist, and therefore $\overline{G}$ is not connected, a contradiction.
  So the $M_i's$ do not overlap.
- To finish the proof, it suffices to notice that the quotient graph is prime, else at least one of the M_i would not be maximal.
- At least the $M_i's$ are strong modules since they partition $V(G)$.

- One can notice, that we use very little of graph theory in this proof.
- This kind of theorem is valid for various combinatorial decompositions of directed graphs, 2-structures, hypergraphs, boolean functions, submodular functions, common intervals to a set of permutations . . .

A general combinatorial decomposition

Fact

The set of all modules of an undirected graph (resp. a directed graph)
constitutes a partitive family (resp. a weak partitive family).

Uniqueness decomposition theorem, Chein Habib Maurer 1981

Partitive (resp. weakly partitive) families admit a decomposition tree with two (resp. three) types of nodes :

- degenerate (also called fragile)
- prime
- (resp. linear)

- This tree representation theorem for partitive (resp. weakly partitive) families $F \subseteq 2^{|X|}$, yields an encoding of these families in $O(|X|)$.
- This kind of combinatorial decomposition of a set family has been generalized in many directions.

Application : Modular Decomposition Theorem for Directed Graphs

Theorem (Chein, Habib, Maurer 1981)

Let $G = (V, E)$ be a directed graph with $|V| \geq 4$, the four following cases are mutually exclusive :

1. *G is not connected, Parallel node*
2. *$\overline{G}$ is not connected, Series node*
3. *G^* is not strongly connected, Linear node*
4. *$G_{/\mathcal{M}(G)}$ is a prime graph, with $\mathcal{M}(G)$ the modular partition containing the maximal strong modules of G, Prime node*

$G*$ est en fait le graphe G modifié comme suit :
A chaque non arête [x,y] de G (les deux arcs xy et yx ne sont pas dans G) on ajoute les deux arcs dans $G*$ les deux arcs xy et yx.
Supposons une décomposition linéaire de G en G(X1).... G(Xk) les arcs entre les G(Xi) ne changent pas dans G* et donc les composantes fortement connexes de G* sont incluses dans les G*(Xi).
Chaque G(Xi) ne peut pas être un noeud linéaire, car on prend la décomposition maximale (dans l'arbre de décomposition il n'y a jamais deux noeuds serie (resp. parallele ou linéaire) adjacents) mais soit :
- G(Xi) est premier mais le graphe des non arêtes est aussi connexe et donc dans G* on a G*(Xi) fortement connexe.
- G(Xi) est un noeud série (complet symétrique) et il est fortement connexe.
-G(Xi) est un noeud parallèle et on le rend fortement connexe dans G* en ajoutant toutes les arcs symétriques entre les composantes.

Prime graphs are nested

Folklore Theorem

Let G be a prime graph ($|G| \geq 4$), then G contains a P_4.

Theorem Schmerl, Trotter, Ille 1991 ...

Let G be a prime graph ($|G| = n \geq 4$), then G contains a prime graph on $n-1$ vertices or a prime graph on $n-2$ vertices.

A simple proof

A stronger statement, Cournier, Ille, 1991

For a prime graph there is at most one vertex not contained in a P_4.

Proof

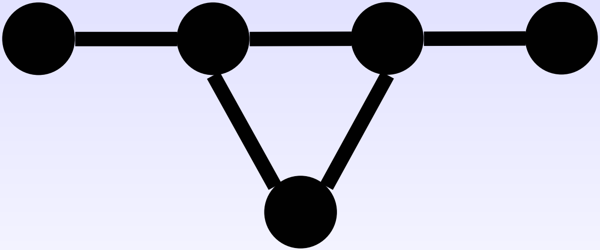
As a prime graph G is necessarily connected and if $\exists x \in V$ that does not belong to a P_4. Every connected component of $\overline{N(x)}$ is a module, therefore $\overline{N(x)}$ must be a stable set.
If $\exists x \neq y \in V$ that does not belong to a P_4,
wlog assume $xy \notin E$
But then $y \in \overline{N(x)}$ and therefore : $N(y) \subseteq N(x)$.
By symmetry x, y must be false twins, a contradiction.

Nota Bene : this result also holds for infinite graphs !
So does our previous proof.

corollary

P_4 free graphs are cographs.

In fact if there is such a vertex $x \in V$,
x is adjacent to the middle vertices of a P_4.
(Such a subgraph is called a **bull**).



A bull is isomorphic to its complement.

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