Cours 3 : Graph Searching is playing with orders MPRI 2016–2017

Michel Habib
habib@liafa.univ-Paris-Diderot.fr
http://www.irif.fr/~habib

27 septembre 2016

Schedule of this course

Introduction to graph search

End vertices

TBLS, a Tie-Breaking Label Search

Two new searches LEXUP and LEXDOWN with no application

Introduction to graph search

End vertices

TBLS, a Tie-Breaking Label Search

Two new searches LEXUP and LEXDOWN with no application

Some definitions

Graph Search

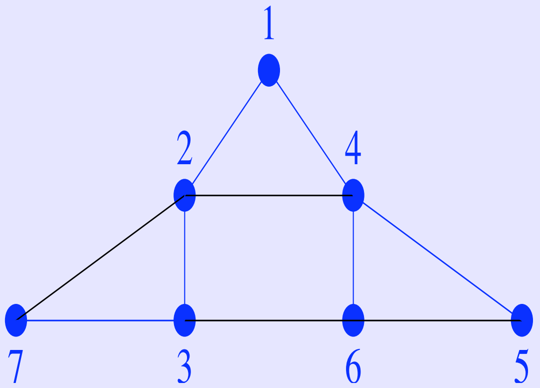
The graph is supposed to be connected so as the set of visited vertices. After choosing an initial vertex, a search of a connected graph visits each of the vertices and edges of the graph such that a new vertex is visited only if it is adjacent to some previously visited vertex.
At any point there may be several vertices that may possibly be visited next. To choose the next vertex we need a tie-break rule.
The breadth-first search (BFS) and depth-first search (DFS) algorithms are the traditional strategies for determining the next vertex to visit.

They are many formalisms to represent graph search depending on the view point :

- Data structures involved
- Implementation issues
- Path algebras
- Bio-inspired graph searches (as used by Amos Korman with ants)
- Reachability problems in complexity theory
- Here we want to focus on the visiting ordering of the vertices and on the tie-break process that distinguishes graph searches

1. For us a graph search just produces a total ordering of the vertices.
2. In the following an ordering of the vertices, always means a total ordering of the vertices.
3. Seminal paper with a systematic study of graph search : D.G. Corneil et R. M. Krueger, A unified view of graph searching, SIAM J. Discrete Math, 22, Num 4 (2008) 1259-1276

Generic Search



Invariant

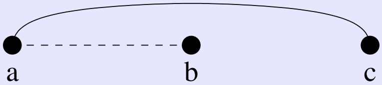
At each step, an edge between a visited vertex and a unvisited one is selected

Generic search

```
S ← {s}
for i ← 1 to n do
    Pick an unnumbered vertex v of S
    σ(i) ← v
    foreach unnumbered vertex w ∈ N(v) do
        if w ∉ S then
            Add w to S
        end
    end
end
```

Generic question ?

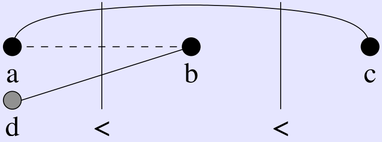
Let a, b et c be 3 vertices such that $ab \notin E$ et $ac \in E$.



Under which condition could we visit first a then b and last c ?

Property (Generic)

For an ordering σ on V, if $a <_\sigma b <_\sigma c$ and $ac \in E$ and $ab \notin E$, then it must exist a vertex d such that $d <_\sigma b$ et $db \in E$



Theorem

For a graph $G = (V, E)$, an ordering σ on V is a generic search of G iff σ satisfies property (Generic).

Most of the searches that we will study are refinement of this generic search

- i.e. we just add new rules to follow for the choice of the next vertex to be visited
- DFS (Stack), BFS(Queue), Dijkstra (Heap), ...
- Graph searches mainly differ by the management of the tie-break set

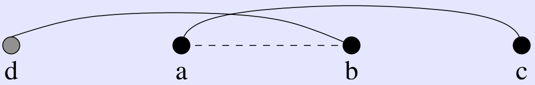
BFS

Data: a graph $G = (V, E)$ and a start vertex $s \in V$
Result: an ordering σ of V
Initialize *queue* to $\{s\}$
for $i \leftarrow 1$ **à** n **do**
    dequeue v from beginning of *queue*
    $\sigma(i) \leftarrow v$
    foreach *unnumbered vertex* $w \in N(v)$ **do**
        if *w is not already in queue* **then**
            enqueue w to the end of *queue*
        end
    end
end

Algorithm 1: Breadth First Search (BFS)

Property (BFS)

For an ordering σ on V, if $a <_\sigma b <_\sigma c$ and $ac \in E$ and $ab \notin E$, then it must exist a vertex d such that $d <_\sigma a$ et $db \in E$



Theorem

For a graph $G = (V, E)$, an ordering σ on V is a BFS of G iff σ satisfies property (BFS).

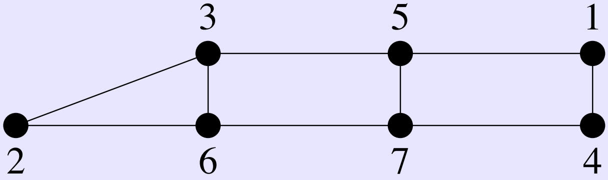
Applications of BFS

1. Distance computations (unit length), diameter and centers, (see course # 2)
2. BFS provides a useful layered structure of the graph
3. Using BFS to search an augmenting path provides a polynomial implementation of Ford-Fulkerson maximum flow algorithm.

Lexicographic Breadth First Search (LBFS)

Data: a graph $G = (V, E)$ and a start vertex s
Result: an ordering σ of V
Assign the label $\emptyset$ to all vertices
$label(s) \leftarrow \{n\}$
for $i \leftarrow n$ **à** 1 **do**
  Pick an unnumbered vertex v with lexicographically largest label
  $\sigma(i) \leftarrow v$
  foreach *unnumbered vertex* w *adjacent to* v **do**
    $label(w) \leftarrow label(w).\{i\}$
  end
end

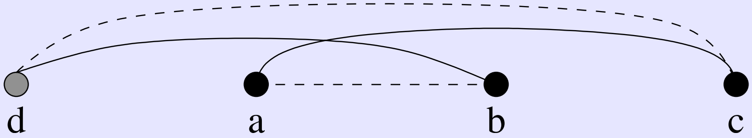
Algorithm 2: LBFS


3
5
1
2
6
7
4

It is just a breadth first search with a tie break rule.
We are now considering a characterization of the order in which a LBFS explores the vertices.

Property (LexB)

For an ordering σ on V, if $a <_\sigma b <_\sigma c$ and $ac \in E$ and $ab \notin E$, then it must exist a vertex d such that $d <_\sigma a$ et $db \in E$ et $dc \notin E$.



Theorem

For a graph $G = (V, E)$, an ordering σ on V is a LBFS of G iff σ satisfies property (LexB).

Why LBFS behaves so nicely on well-structured graphs

A nice recursive property

On every tie-break set S, LBFS operates on $G(S)$ as a LBFS.

proof

Consider $a, b, c \in S$ such that $a <_\sigma b <_\sigma c$ and $ac \in E$ and $ab \notin E$, then it must exist a vertex d such that $d <_\sigma a$ et $db \in E$ et $dc \notin E$. But then necessarily $d \in S$.

Remark

Analogous properties are false for other classical searches.

LexBFS versus LBFS !

Google Images query : LBFS (thanks to Fabien)

yields :

One of the First Answer

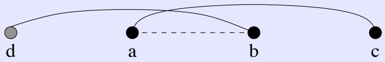
Applications of LBFS

1. Most famous one : chordal graph recognition via simplicial elimination schemes (easy application of the 4-points condition)
2. For many classes of graphs using LBFS ordering "backward" provides structural information on the graph.
3. Last visited vertex (or clique) has some property (example simplicial for chordal graph)
4. Of course property LexB was known by authors such as Tarjan or Golumbic to study chordal graphs but they did not noticed that it was a characterization of LBFS.

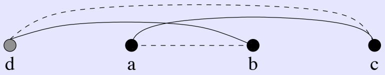
LDFS

BFS vs LBFS

BFS

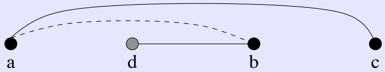


LBFS

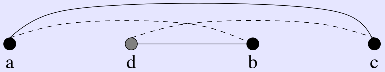


DFS vs LDFS

DFS

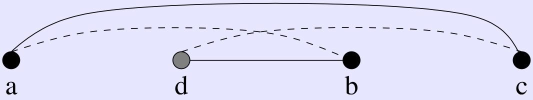


LDFS



Property (LD)

For an ordering σ on V, if $a <_\sigma b <_\sigma c$ and $ac \in E$ and $ab \notin E$, then it must exist a vertex d such that $a <_\sigma d <_\sigma b$ and $db \in E$ and $dc \notin E$.



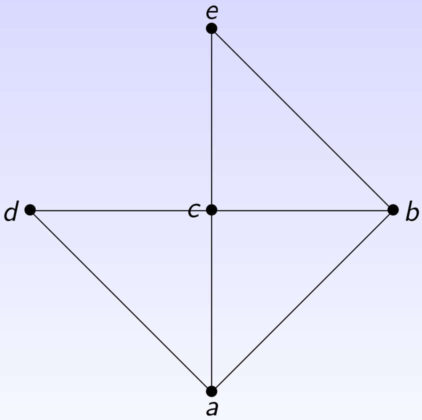
Theorem

For a graph $G = (V, E)$, an ordering σ on V is a LDFS of G iff σ satisfies property (LD).

Lexicographic Depth First Search (LDFS)

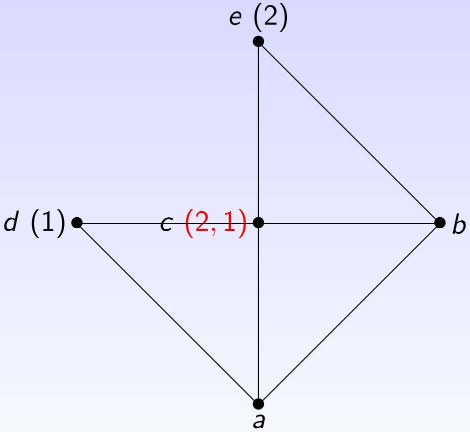
Data: a graph $G = (V, E)$ and a start vertex s
Result: an ordering σ of V
Assign the label $\emptyset$ to all vertices
$label(s) \leftarrow \{0\}$
for $i \leftarrow 1$ **à** n **do**
  Pick an unnumbered vertex v with lexicographically largest label
  $\sigma(i) \leftarrow v$
  foreach *unnumbered vertex* w *adjacent to* v **do**
    $label(w) \leftarrow \{i\}.label(w)$
  end
end

LDFS example



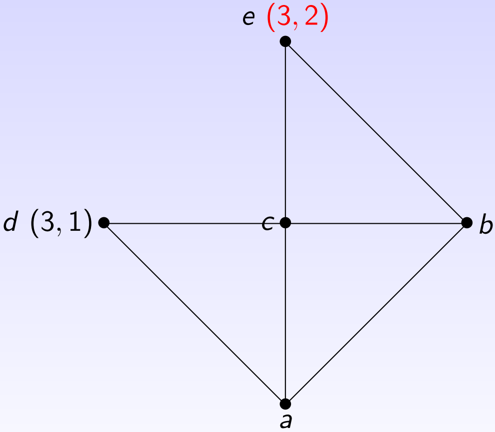
LDFS visiting *a* then *b*

LDFS example II



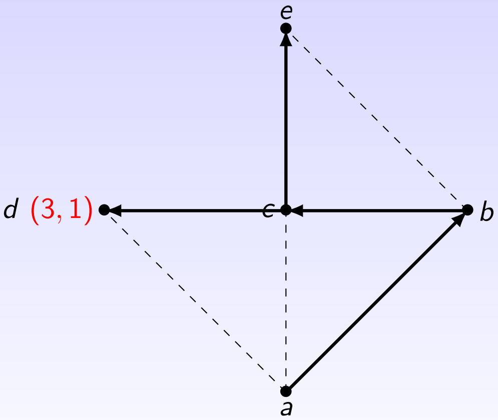
LDFS visiting a then b must visit c

LDFS example III



LDFS visiting a, b, c must visit e

LDFS example IV



LDFS visiting a, b, c, e and must finish in d

Applications of LDFS

- Hard to find application of this new tool !
- Finding long paths, a very simple greedy algorithm :
- *LDFS-based certifying algorithm for the Minimum Path Cover problem on cocomparability graphs*
  D. Corneil, B. Dalton and M. Habib **SIAM J. of Computing** 42(3) : 792-807 (2013).
- Nice graph based heuristics for community detection, joint work with J. Creusefond (PhD in Caen).

- So far we have considered visiting orderings of the vertices which characterize graph searches such as Generic Search, BFS, DFS, LBFS, LDFS.
- These orderings allow to prove properties on graph searches (without considering the algorithm itself)
- But also to certify (as for example for BFS and diameter computations)
- Let us now go a little further with tie-breaking.

Introduction to graph search

End vertices

TBLS, a Tie-Breaking Label Search

Two new searches LEXUP and LEXDOWN with no application

End-vertex problem for a search S :
Input : A graph $G = (V, E)$, and a vertex t.
Question : Is there σ an S-ordering of V such that $\sigma(n) = t$?

Theorem

Given a bipartite graph G and a vertex v of G, it is NP-complete to decide if there exists an execution of BFS on G such that v is the end-vertex.

Reduction direct from 3-SAT

For every $n \in \mathcal{N}$ we define a graph G_n, which has one special vertex r_n called the root. It is constructed recursively as follows :

- G_0 is the graph with one vertex r_0.
- G_n is constructed from G_{n-1} by first adding three vertices : the new root r_n, and its two neighbours y_n and $\overline{y_n}$, that are also adjacent to r_{n-1}. Finally we attach a path of $2n-1$ new vertices to y_n (respectively $\overline{y_n}$) and label its end-vertex x_n (respectively $\overline{x_n}$).

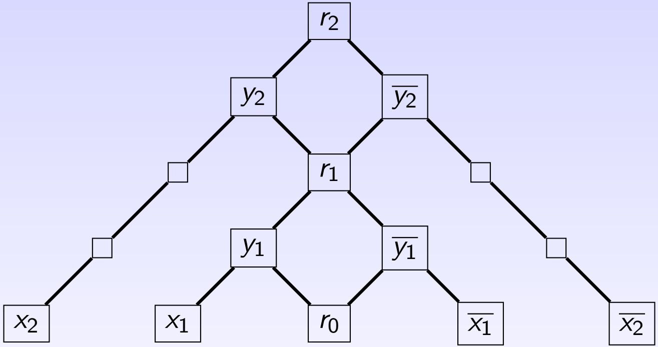


FIGURE: The graph G_2.

Proposition 1

G_n is a bipartite graph that has $(2n+1)(n+1)$ vertices that all are at distance at most $2n$ from r_n. There are $2n+1$ vertices at distance exactly $2n$ from r_n, and these are $x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots, x_n, \overline{x_n}$ and r_0.

The following proposition is central to the reduction and concerns the order that we obtain on those $2n+1$ vertices when we do a BFS starting at the root.

Proposition 2

Consider an order on the vertices of G_n given by an execution of a BFS starting at r_n. For each $1 \leq i \leq n$ at most one of x_i and $\overline{x_i}$ is before r_0. Moreover each of the 2^n choices of one among x_i and $\overline{x_i}$ for each i, can be obtained as the set of vertices that appear before r_0 for some BFS order of G_n.

End of the proof

1. We just add to the previous gadjet, the incidence bipartite clauses–variables and a pending edge $r_0 t$.
2. There is a BFS ending at t iff the SAT instance is satisfiable.

End-vertex results	BFS	LBFS	DFS	LDFS	MNS
All Graphs	**NPC**	NPC	**NPC**	**NPC**	?(P)
Bipartite	**NPC**	?(NPC)	?(NPC)	?(NPC)	?(P)
Weakly Chordal	**NPC**	NPC	**NPC**	**NPC**	?(P)
Chordal	?(NPC)	?(NPC)	**NPC**	?	P
Split	**P**	**P**	**NPC**	**P**	P
Str. Chordal Split	**P**	**P**	**NPC**	**P**	P
Path Graphs	?	?(P)	**NPC**	?	P

Introduction to graph search

End vertices

TBLS, a Tie-Breaking Label Search

Two new searches LEXUP and LEXDOWN with no application

A *graph search* is an iterative process that chooses at each step a vertex of the graph and numbers it (from 1 to n). Each vertex is chosen (also said *visited*) exactly once (even if the graph is disconnected). Let us now define a *General Tie-Breaking Label Search* (TBLS). It uses *labels* to decide the next vertex to be visited ; $label(v)$ is a subset of $\{1, \ldots, n\}$. A TBLS is defined on :

1. A graph $G = (V, E)$ on which the search is performed ;
2. A strict partial order $\prec$ over the label-set $P(\mathbb{N}^+)$;
3. An ordering τ of the vertices of V called the *tie-break permutation*.

TBLS$(G, \prec, \tau)$

foreach $v \in V$ **do** $label(v) \leftarrow \emptyset$;
for $i \leftarrow 1$ **to** n **do**
    $Eligible \leftarrow \{x \in V \mid x$ unnumbered and $\nexists y \in V$ such that $label(x) \prec label(y)\}$;
    Let v be the leftmost vertex of *Eligible* according to the ordering τ;
    $\sigma(v) \leftarrow i$;
    foreach unnumbered vertex w adjacent to v **do**
        $label(w) \leftarrow label(w) \cup \{i\}$;
    end
end

The ideas of this formalism :

- A graph search just produces a vertex ordering
- Any ordering of the vertices can be used as a tie-break
- Then we can iterate graph searches and study what orderings can be obtained

With this formalism, in order to specify a particular search we just need to specify $\prec$, the partial order relation on the label sets for that search. The choice of permutation τ is useful in some situations described below ; otherwise, we consider the orderings output by an arbitrary choice of τ thanks to the following definition :

Definition

Let $\prec$ be some ordering over $P(\mathbb{N}^+)$. Then σ is a TBLS ordering for G and $\prec$ if there exists τ such that $\sigma = TBLS(G, \prec, \tau)$.

Usual conventions

$\mathbb{N}^+$ represents the set of integers strictly greater than 0 and $\mathbb{N}^+_p$ represents the set of integers strictly greater than 0 and less than p. $P(\mathbb{N}^+)$ denotes the power-set of $\mathbb{N}^+$ and $\mathfrak{S}_n$ denotes the set of all permutations of $\{1, ..., n\}$.
We always use the notation $<$ for the usual strict (i.e., irreflexive) order between integers, and $\prec$ for a partial strict order between elements from $P(\mathbb{N}^+)$ (or from another set when specified).

So far in the model we play with :

a number (label) associated to each vertex
a set a labels associated to each unnumbered vertex
and some partial order $\prec$ between sets of labels

A useful property

Theorem

A graph G, a search rule $\prec$, σ an ordering, then :
There exists τ such that $\sigma = TBLS(G, \prec, \tau)$ iff
$\sigma = TBLS(G, \prec, \sigma)$

Generic Search

We define $A \prec_{gen} B$ if and only if $A = \emptyset$ and $B \neq \emptyset$ and let σ be a permutation of V. The following conditions are equivalent :

1. σ is a generic search ordering of V (a TBLS using $\prec_{gen}$).
2. For every triple of vertices $a,\ b,\ c$ such that $a <_\sigma b <_\sigma c$, $a \in N(c) - N(b)$ there exists $d \in N(b)$ such that $d <_\sigma b$.

Ad-hoc min and max operators

For $A \in P(\mathbb{N}^+)$,

- let $umin(A)$ be :
  if $A = \emptyset$ then $umin(A) = \infty$ else $umin(A) = min\{i | i \in A\}$;
- and $umax(A)$ be :
  if $A = \emptyset$ then $umax(A) = 0$ else $umax(A) = max\{i | i \in A\}$.

DFS

We define $A \prec_{DFS} B$ if and only if $umax(A) < umax(B)$. Let σ be a permutation of V. The following conditions are equivalent :

1. σ is a DFS ordering (a TBLS using $\prec_{DFS}$).
2. for every triple of vertices $a,\ b,\ c$ such that $a <_\sigma b <_\sigma c$, $a \in N(c) - N(b)$ there exists $d \in N(b)$ such that $a <_\sigma d <_\sigma b$.
3. for every triple of vertices $a,\ b,\ c$ such that $a <_\sigma b <_\sigma c$, and a is the rightmost vertex of $N(b) \cup N(c)$ in σ, we have $a \in N(b)$.

BFS

We define $A \prec_{BFS} B$ if and only if $umin(A) > umin(B)$. Let σ be a permutation of V. The following conditions are equivalent :

1. σ is a BFS ordering (a TBLS using $\prec_{BFS}$).
2. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$, $a \in N(c) - N(b)$, there exists d such that $d \in N(b)$ and $d <_\sigma a$.
3. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$ and a is the leftmost vertex of $N(b) \cup N(c)$ in σ, we have $a \in N(b)$.

LDFS

We define $A \prec_{LDFS} B$ if and only if $umax(A - B) < umax(B - A)$.
Let σ be a permutation of V. The following conditions are equivalent :

1. σ is a LDFS ordering (a TBLS using $\prec_{LDFS}$)
2. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$, $a \in N(c) - N(b)$, there exists $a <_\sigma d <_\sigma b$, $d \in N(b) - N(c)$.
3. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$ and a is the rightmost vertex in $N(b) \triangle N(c)$ in σ, $a \in N(b) - N(c)$.

LBFS

We define $A \prec_{LBFS} B$ if and only if $umin(B - A) < umin(A - B)$. Let σ be a permutation of V. The following conditions are equivalent :

1. σ is a LBFS ordering (a TBLS using $\prec_{LBFS}$)
2. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$, $a \in N(c) - N(b)$, there exists $d <_\sigma a$, $d \in N(b) - N(c)$.
3. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$ and a is the leftmost vertex of $N(b) \triangle N(c)$ in σ, then a $\in N(b) - N(c)$.

LDFS

We define $A \prec_{LDFS} B$ if and only if $umax(A - B) < umax(B - A)$. Let σ be a permutation of V. The following conditions are equivalent :

1. σ is a LDFS ordering (a TBLS using $\prec_{LDFS}$)
2. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$, $a \in N(c) - N(b)$, there exists $a <_\sigma d <_\sigma b$, $d \in N(b) - N(c)$.
3. for every triple $a, b, c \in V$ such that $a <_\sigma b <_\sigma c$ and a is the rightmost vertex in $N(b) \triangle N(c)$ in σ, $a \in N(b) - N(c)$.

Consequences

- In our model only $\prec$ matters for a graph search.
- Nice mathematical duality min-max between BFS and DFS (resp. LBFS and LDFS).
- A stack (resp. a queue) is a data structure to manage a Maximum (resp. Minimum) current value.
- But which data structures to implement $max(A \setminus B)$ or $min(A \setminus B)$?
- New characterizations of their orderings
- Prove results just using the orderings (without looking at all at the implementation).

Repeated LBFS$^+$ in this new model

Data: G an undirected graph, σ_0 a permutation on $V(G)$
Result: an ordering $\sigma_{|V(G)|}$
for $i = 1$ **to** $|V(G)|$ **do**
  $\sigma_i \leftarrow TBLS(G, \prec_{LBFS}, \sigma_{i-1}^d)$;
end
Output $\sigma_{|V(G)|}$;

A semi-lattice structure

Definition

For two TBLS searches S, S', we say that S' is an extension of S (denoted by $S \ll S'$) if and only if every S'-ordering σ also is an S-ordering. [a]

---

a. This definition is consistent with the usual extension ordering used in partial order theory ; in particular $S \ll S'$ means that the set of comparabilities in S is included in those of S'.

Definition

For two partial orders $\prec_P, \prec_Q$ on the same ground set X, we say that P extends Q if $\forall x, y \in X$, $x \prec_Q y$ implies $x \prec_P y$.

Theorem

Let S, S' be two TBLS. S' is an extension of S if and only if $\prec_{S'}$ is an extension of $\prec_S$.

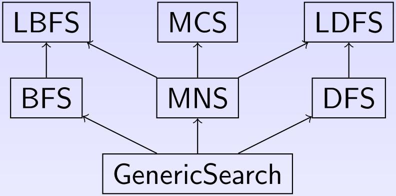


FIGURE: Summary of the heredity relationships. An arc from Search S to search S' means that S' extends S.

Clearly being an extension is a transitive relation. In fact $\ll$ structures the TBLS graph searches as $\wedge$-semilattice. The 0 search in this semi-lattice, denoted by the null search or S_{null}, corresponds to the empty ordering relation (no comparable pairs). At every step of S_{null} the Eligible set contains all unnumbered vertices. Therefore for every τ, $TBLS(G, \prec_{S_{null}}, \tau) = \tau$ and so any total ordering of the vertices can be produced by S_{null}.

The infimum between two searches S, S' can be defined as follows : For every pair of label sets A, B, we define : $A \prec_{S \wedge S'} B$ if and only if $A \prec_S B$ and $A \prec_{S'} B$.

Exercise

Show that Layered Search does not belong to the TBLS formalism.
How to include it ?

LEXUP

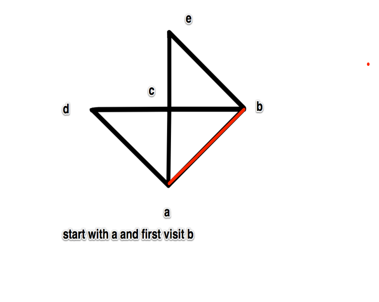
Data: a graph $G = (V, E)$ and a start vertex s;
Result: an ordering σ of V;
Assign the label $\emptyset$ to all vertices ;
$label(s) \leftarrow \{n\}$;
for $i \leftarrow 1$ **à** n **do**
    Pick an unnumbered vertex v with lexicographically largest label;
    $\sigma(i) \leftarrow v$;
    foreach *unnumbered vertex* w *adjacent to* v **do**
        $label(w) \leftarrow label(w).\{i\}$;
    end
end

Algorithm 3: LEXUP

LEXDOWN

Data: a graph $G = (V, E)$ and a start vertex s;
Result: an ordering σ of V;
Assign the label $\emptyset$ to all vertices ;
$label(s) \leftarrow \{n\}$;
for $i \leftarrow n$ **à** 1 **do**
    Pick an unnumbered vertex v with lexicographically largest label;
    $\sigma(i) \leftarrow v$;
    foreach *unnumbered vertex* w *adjacent to* v **do**
        $label(w) \leftarrow \{i\}.label(w)$;
    end
end

Algorithm 4: LEXDOWN



$LBFS$: a, b, c, d, e

$LDFS$: a, b, c, e, d

$LEXUP$: a, b, e, c, d Hamilton path ! ! !

$LEXDOWN$: a, b, d, c, e

All different orderings in this simple example.

Potential application to AI discovery algorithms.

Perspectives

- Use this tie-break model to prove properties of graph searches
- Understanding the greedy aspects of LBFS and LDFS on cocomparability graphs.
- More precisely : matroidal aspects for LDFS and anti-matroidal for LBFS.
- Study iterations of graph searches and their cycles.
- Find applications of 2 new lexicographic searches (Lexup, Lexdown) which came by symmetry on the algorithms.

- *Influence of the tie-break rule on the end-vertex problem*
  Pierre Charbit, Michel Habib, Antoine Mamcarz
  Discrete Mathematics & Theoretical Computer Science, Vol 16, No 2 (2014).

  `http://www.dmtcs.org/dmtcs-ojs/index.php/dmtcs/article/view/2519`
- TBLS : *A tie-break model for graph search*
  with Derek Corneil, Jérémie Dusart, Michel Habib, Antoine Mamcarz and Fabien de Montgolfier
  à paraître dans **Discrete Applied Mathematics**.
- Informations about LEXUP and LEXDOWN in J. Dusart's PhD, june 2014.