Examen: MPRI Algorithmique de graphes

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NB : * indicates hard questions.

1 Chordal graphs

In this problem we consider a connected chordal graph G (with n vertices and m edges).

1. Show that every chordal graph can be represented as the intersection graph of a family of subtrees in a host tree with maximum degree 3.
2. Let S be a minimal separator in G, show that it always exist two different maximal cliques C, C' of G such that :
   $C \cap C' = S$ and $\forall a \in C - C'$, $\forall b \in C' - C$, S is a minimal separator for a and b.
3. * Show that every minimal separator S belongs to all maximal clique tree of G (i.e. every maximal clique tree T contains an edge labelled with S).
4. Show that each maximal clique tree uses exactly the same collection of minimal separators, including repetitions (i.e. if a separator S labels α edges of some maximal clique tree, then every maximal clique tree contains α edges labelled with S).
   Give examples of families of graphs for which the number of repetitions of a given minimal separator is unbounded in any maximal clique tree.
5. If we consider the edges of the clique tree labelled with the size of the minimal separators, show that : for every maximal clique tree T
   $weight(T) = \Sigma_{1 \leq i \leq k} |C_i| - n$, where $C_1, \ldots C_k$ are the maximal cliques of G.
6. Prove that a graph G is a forest iff every pairwise intersecting family of paths in G has a common vertex (i.e. family of paths satisfy Helly's property).

1.1 Distance properties on chordal graphs

1. A convex structure consists of a set X together with a collection $\mathcal{C}$ of subsets of X (the convex sets) such that : the empty set and X are convex and the intersection of convex sets is convex.
   For a graph $G = (V, E)$ the usual convexity is defined as follows :
   A subset $S \subseteq V$ is convex if for every two vertices $x, y \in S$, all vertices on shortest paths between x and y are also contained in S.
   Show that this correctly defines a convexity.

2. Show that we can obtain a new notion of convexity called chordless-convexity if we replace in the above definition "shortest path" by "chordless path". Are these two definitions of convexity equivalent ?
3. Show that $G = (V, E)$ is chordal iff $\forall v \in V \ N[v] = N(v) \cup \{v\}$ is chordless convex.
4. * An extreme point of a convex set S is a point $x \in S$ such that $S - x$ is still convex. The convex hull of a given subset $A \subseteq X$ is the smallest convex set containing A.

   Show that for chordal graphs chordless convexity defines a convex geometry (i.e. every convex set is the convex hull of its extreme points).
5. For a graph G, connected but not necessarily chordal, let $C(G)$ be the induced subgraph of G made up with its centers.

   Shows that this graph is not always connected.
6. * For a chordal graph G show that :

   $C(G)$ is connected and $diam(C(G)) \leq 3$
7. ** Propose an efficient algorithm to compute a center (resp. all the centers) of a chordal graph.

2 Algorithms

1. Propose a linear time algorithm to recognize co-chordal graphs (i.e. complement of chordal graphs).
2. **Maximal cardinality Search (MCS)**

```
Data: a graph G = (V, E) and a start vertex s
Result: an ordering σ of V
Assign the label 0 to all vertices
label(s) ← 1
for i ← n à 1 do
    Pick an unnumbered vertex v with largest label
    σ(i) ← v
    foreach unnumbered vertex w adjacent to v do
        label(w) ← label(w) + 1
    end
end
```

   Propose a linear time implementation of MCS algorithm.
3. Is-it possible to modify (by adding a test) MCS Algorithm in order to check if the graph is chordal (without testing that σ provides a simplicial elimination ordering) ?

3 Treewidth, branchwidth

We have seen during the course when proving $treewidth(G) \leq 3/2 branchwidth(G)$: that a decomposition tree for treewidth can be transformed in a branchwidth like tree.

Does there exist a way to express treewidth in terms of a cost function on a ternary tree ? Detail the cost function if any.