Graph Algorithms
Exam : Partiel 1

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Notations

For an undirected simple (no loops, no multiple edges) finite graph G, we denote by $V(G)$ its finite set of vertices and by $E(G)$ the finite set of edges. All the graphs are supposed to be connected.
NB : questions are mostly independent, * means a question that could be difficult.

I) Separable graphs

A *separable* graph is a graph G in which the vertices can be partitioned into a clique $C(G)$ and an independent set $I(G)$.

1. Show that every separable graph is chordal.
2. Show that if G is a separable graph then also its complement $\overline{G}$ is a separable graph.
3. Prove the following equivalence :
   G is a separable graph iff it can be represented as the intersection of substars in a star graph ($K_{1,n}$).
   Hint : consider the structure of maximal clique trees.
4. Show that for a non complete separable graph G : $diam(G) \in \{2, 3\}$, $radius(G) \in \{1, 2\}$ and $radius(G) = 1$ implies $diam(G) = 2$.
5. Propose a linear time algorithm to compute the radius of a separable graph.
6. Show examples of separable graphs having radius 2, but diameter 2 and 3.
7. * Can you propose a linear time algorithm for computing the diameter of separable graphs ?

II) Graph Searches

1. For a connected graph G, when a BFS (resp. DFS) ordering σ of $V(G)$ is given, how to compute the associated rooted tree of the search. This rooted tree can de defined using the standard *parent* function.
2. Show that if $M \subseteq V(G)$ is a module of G and σ a LBFS ordering of $V(G)$ then $\sigma(M)$ the induced ordering of M induced by σ is a legitimate LBFS ordering on $G(M)$ the subgraph induced by M.
3. Does this hold also for BFS, DFS and LDFS ?
4. If $\mathcal{P}$ is a modular partition of $V(G)$ and σ a LBFS ordering of $V(G)$, show that ordering the elements of $\mathcal{P}$ by their discovery time with σ gives a legitimate LBFS ordering of $G/\mathcal{P}$.
5. * Let G be a prime connected undirected graph, and σ a LBFS ordering of G starting at x and ending at y.
   Show that for every vertex z such that $x <_\sigma z <_\sigma y$, there exists a path form z to x avoiding $N(y) \cup \{y\}$.
6. * Recall the 4-points condition characterizing LBFS orderings. Show that it can be generalized as follows :
   Let us consider σ a LBFS ordering on a graph G and let z be the last vertex of this ordering. For all triple of vertices, $a, b, c \in V(G)$ such that $c <_\sigma b <_\sigma a$ and $ca \in E(G)$, there exists a path from b to c whose internal vertices are disjoint from $N(z) \cup \{z\}$.

III) Cocomparability graphs

Let us start with some definitions :

- A *cocomparability* graph is an undirected graph whose complement can be transitively oriented. To orient an undirected graph G for each edge $xy \in E(G)$, we choose an arc from x to y or from y to x or both.
- An *umbrella* for a total ordering σ of $V(G)$ is a triple of vertices (x, y, z) such that : $x <_\sigma y <_\sigma z$ with $xz \in E(G)$ and $xy, yz \notin E(G)$.
- A *cocomp* ordering of a graph G is a total ordering of the vertices with no umbrella.

1. Show that if an undirected graph admits a transitive orientation it also admits an acyclic transitive orientation (i.e., without directed cycle).

   Hint : consider the strongly connected components of a transitive orientation having directed cycles.

   From now on transitive orientation stands for acyclic transitive orientation.
2. Show that G is a cocomparability graph iff it admits a cocomp ordering.
3. A *cobipartite* graph is simply the complement of a bipartite graph. Show that every cobipartite graph is a cocomparability graph.
4. Propose your best algorithm to check if an ordering is a cocomp ordering. What is the worst case time complexity of your algorithm ?
5. * Show that for every cocomparability graph G there always exists a LBFS ordering which is also a cocomp ordering.
6. Let $\sigma = x_1, \ldots x_n$ be a LBFS cocomp ordering of a connected graph G, show that the unique path μ from x_1 to x_n used in the LBFS is a dominating path of G (i.e., every vertex of $V(G)$, either belongs to μ or is adjacent to μ).

IV) Putting things together

1. * Let σ be a LBFS ordering of cocomparability graph G, prove that the last vertex of σ can be taken as a source (or a sink) in a transitive orientation of $\overline{G}$.

   Hint : first consider the case where G is a prime graph.
2. * How to use this fact and design an algorithm that computes a transitive orientation of $\overline{G}$?