INEQUALITIES FOR ONE-STEP PRODUCTS

MÁRIO J. J. BRANCO AND JEAN-ÉRIC PIN

Abstract. Let a be a letter of an alphabet A. Given a lattice of languages $\mathcal{L}$, we describe the set of ultrafilter inequalities satisfied by the lattice $\mathcal{L}_a$ generated by the languages of the form L or LaA^*, where L is a language of $\mathcal{L}$. We also describe the ultrafilter inequalities satisfied by the lattice $\mathcal{L}_1$ generated by the lattices $\mathcal{L}_a$, for $a \in A$. When $\mathcal{L}$ is a lattice of *regular* languages, we first describe the profinite inequalities satisfied by $\mathcal{L}_a$ and $\mathcal{L}_1$ and then provide a small basis of inequalities defining $\mathcal{L}_1$ when $\mathcal{L}$ is a Boolean algebra of regular languages closed under quotient.

The *concatenation product* of languages and its connection with algebra and logic has been a very active research area over the past fifty years. It is often sufficient to consider *one-step products* of the form $L \to LaA^*$ where a is a letter of the alphabet A, or their dual forms $L \to A^*aL$. For instance, it has been shown [18, 39, 40] that a variety of regular languages closed under these two operations is also closed under product. It is also known that a regular language belongs to the smallest variety of languages closed under one-step products if and only if its syntactic monoid is $\mathcal{R}$-trivial. One step products were also used in [37] to describe the languages whose syntactic monoid is idempotent (see also [9, 11]) and in [10] to get the expressive power of linear temporal logic without until.

The purpose of this article is to conduct a comprehensive study of one-step products, first for arbitrary languages, then for regular languages, using the so called *equational approach*.

Historical background. In the regular case, the equational approach goes back to Schützenberger's characterization of star-free languages by the profinite equation $x^{\omega+1} = x^\omega$ [32]. Two results make it possible to account for similar situations: Eilenberg's variety theorem [11], which gives a bijection between varieties of regular languages and varieties of finite monoids and Reiterman's theorem [29] which provides a description of varieties of finite monoids by profinite equations.

During the years 1975–2000, much effort was devoted to *operations on regular languages*, notably concatenation product [35, 26]. One-step products were first considered as an exercise in Eilenberg's book [11, Exercise IX. 2.1] and a deep result of [4] led to an equational characterization of this operation in the variety setting.

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However, varieties of languages soon proved to be an overly constrained concept and a series of generalizations were successively introduced [19, 12, 36, 25, 13], each of them leading to an update of the equational approach. The first of these updates [19] consisted in replacing profinite equations by profinite inequalities and the last one led to a very concise statement: *every lattice of regular languages can be defined by a set of profinite inequalities.*

On the other hand, an even more ambitious generalization was proposed in [14]. It applies to *arbitrary* languages, but the price to pay is to replace profinite words by ultrafilters. Still, a similar result holds: *every lattice of languages can be defined by a set of ultrafilters inequalities.*

Main results. Let A be a finite alphabet, let $\mathcal{L}$ be a lattice of languages of A^* and let a be a letter of A. Let $\mathcal{L}_a$ be the lattice generated by the languages of the form L or LaA^*, where $L \in \mathcal{L}$. Let also $\mathcal{L}_1$ be the lattice generated by the union of all lattices $\mathcal{L}_a$, for $a \in A$.

Our first main result (Theorem 4.1 and Corollary 4.2) gives ultrafilter inequalities defining $\mathcal{L}_a$ and $\mathcal{L}_1$, given the ultrafilter inequalities defining $\mathcal{L}$. A similar result was given in [16] but our inequalities are simpler and have the advantage to give immediately profinite inequalities in the regular case (Theorem 4.3 and Corollary 4.4). Moreover, our approach is quite generic and could easily be transposed to other settings than one-step products.

Our second main result gives a much smaller basis of profinite inequalities when $\mathcal{L}$ is a Boolean algebra of regular languages closed under quotients (Theorem 5.2): *The lattice $\mathcal{L}_1$ admits as a base the set of profinite inequalities of the form $zx = zx^2$, $zxy = zyx$ and $z \leqslant zx$, where x, y and z are profinite words such that the profinite equations $z = zx = zy$ hold in $\mathcal{L}$.* The proof relies on the conjunction of two advanced tools, the derived category of a relational morphism and Simon's theorem on the free category on a finite graph, supplemented by a compactness argument.

Our paper is organized as follows. Section 1 gathers the needed topological notions. Section 2 presents the inequality theory for languages. One step products are introduced in Section 3, and Section 4 provides ultrafilter inequalities for $\mathcal{L}_a$ and $\mathcal{L}_1$. Section 5 gives a base of profinite inequalities for $\mathcal{L}_1$ when $\mathcal{L}$ is a Boolean algebra closed under quotients.

1. Stone duality and inequalities

In this paper, we denote by S^c the complement of a subset S of a set E. We also denote $\overline{L}$ the topological closure of a subset L of a topological space.

Let A be a finite alphabet. A *lattice of languages* is a set $\mathcal{L}$ of languages of A^* closed under finite unions and finite intersections. A lattice closed under complement is a *Boolean algebra*. It is *closed under quotients* if, for each $L \in \mathcal{L}$ and $u \in A^*$, the languages $u^{-1}L$ and Lu^{-1} are also in $\mathcal{L}$. Recall that $u^{-1}L = \{x \in A^* \mid ux \in L\}$ and $Lu^{-1} = \{x \in A^* \mid xu \in L\}$.

Let $\mathcal{B}$ be a Boolean algebra of languages of A^*. An *ultrafilter* of $\mathcal{B}$ is a non-empty subset γ of $\mathcal{B}$ such that:

(1) the empty set does not belong to γ,
(2) if $K \in \gamma$ and $K \subseteq L$, then $L \in \gamma$ (closure under extension)

(3) if $K, L \in \gamma$, then $K \cap L \in \gamma$ (closure under intersection),
(4) for every $L \in \mathcal{B}$, either $L \in \gamma$ or $L^c \in \gamma$ (ultrafilter condition).

Stone duality tells us that $\mathcal{B}$ has an associated compact Hausdorff space $S(\mathcal{B})$, called its *Stone space*. This space is given by the set of ultrafilters of $\mathcal{B}$ with the topology generated by the basis of clopen sets of the form $\{\gamma \in S(\mathcal{B}) \mid L \in \gamma\}$, where $L \in \mathcal{B}$.

Only two Stone spaces are considered in this paper. The first one is the Stone space of the Boolean algebra of all the subsets of A^*, known as the *Stone-Čech compactification* of A^* and denoted by βA^*. An important property of βA^* is that every map f from A^* to a compact space K has a unique continuous extension $\beta f : \beta X \to K$.

The second one is the Stone space of the Boolean algebra of all *regular* subsets of A^*. It was proved by Almeida [1] to be equal to the *free profinite monoid* on A, denoted by $\widehat{A^*}$. Its elements are called *profinite words*. We refer to [2, 20, 21] for more information on this space, which can also be seen as the completion of the metric space (A^*, d), where d is the *profinite metric* on A^*.

Two other facts will be used in this paper. First, if $X, Y \subseteq \widehat{A^*}$, then $\overline{XY} = \overline{X}\,\overline{Y}$. Secondly, the monoid $\widehat{A^*}$ is *equidivisible* [3]. A monoid M is equidivisible if for every $u, v, x, y \in M$, the equality $uv = xy$ implies that there is $t \in M$ such that $ut = x$ and $v = ty$, or such that $xt = u$ and $y = tv$. Consequently, usual definitions on words (prefixes, suffixes, factors) extend to profinite words. This is a crucial difference with βA^*, which is not even a monoid.

Hyperspace. Let X be a Hausdorff space and let $\mathcal{C}(X)$ be the set of its closed subsets[1], called the *hyperspace* of X. For each open set U, let us set

$$U^- = \{C \in \mathcal{C}(X) \mid C \cap U \neq \emptyset\} \qquad U^+ = \{C \in \mathcal{C}(X) \mid C \subseteq U\}.$$

The *Vietoris topology* on $\mathcal{C}(X)$ has as a subbase all the sets of the form U^+ or U^-, where U is open [5, p. 47]. It is known that, equipped with the Vietoris topology, $\mathcal{C}(X)$ is always a compact space.

When X is a metric space, then $\mathcal{C}(X)$ is also a metric space. The metric defining the Vietoris topology of the $\mathcal{C}(\widehat{A^*})$ was explicitly given in [24].

2. Inequalities on languages

The inequality theory for languages was first introduced in [14] and later used in [15, 16, 22]. It is based on the following definitions.

Let $\mathcal{B}$ be a Boolean algebra of languages of A^*, which, in this paper, will either be the set of all languages or the set of all regular languages of A^*.

Definition 2.1. *Let μ_0, μ_1 be ultrafilters of $\mathcal{B}$. A language L of $\mathcal{B}$ satisfies the ultrafilter inequality $\mu_0 \leqslant \mu_1$ if the condition $L \in \mu_0$ implies $L \in \mu_1$, or, equivalently, if the condition $\mu_0 \in \overline{L}$ implies $\mu_1 \in \overline{L}$.*

When $\mathcal{B}$ is the lattice of all regular languages, ultrafilters are profinite words and we use the terms *profinite inequality* and *profinite equation*.

[1] Contrary to a frequent convention, we do include the empty set in $\mathcal{C}(X)$.

Definition 2.1 can be extended to sets of languages and to sets of inequalities. Given a subset $\mathcal{S}$ of $\mathcal{B}$ and an ultrafilter inequality $\mu_0 \leqslant \mu_1$, we say that $\mathcal{S}$ *satisfies the ultrafilter inequality* $\mu_0 \leqslant \mu_1$ (notation $\mu_0 \leqslant_{\mathcal{S}} \mu_1$) to mean that every language of $\mathcal{S}$ satisfies $\mu_0 \leqslant \mu_1$. Thus $\mu_0 \leqslant_{\mathcal{S}} \mu_1$ if and only if $\mu_0 \cap \mathcal{S} \subseteq \mu_1 \cap \mathcal{S}$.

Definition 2.2. *Let E be a set of ultrafilter inequalities. A subset of $\mathcal{B}$* satisfies E *if it satisfies every inequality of E. The set of languages* defined by E *is the set of all languages satisfying E.*

The following result is a consequence of Stone duality, see [14, Theorem 5.1] or [22, Theorem 8.3].

Theorem 2.3. *A subset $\mathcal{L}$ of $\mathcal{B}$ is a sublattice of $\mathcal{B}$ if and only if it can be defined by a set of ultrafilter inequalities of the form $\mu_0 \leqslant \mu_1$, where μ_0 and μ_1 are ultrafilters of $\mathcal{B}$.*

It is convenient to write $\mu_0 = \mu_1$ as a shortcut for $\mu_0 \leqslant \mu_1$ and $\mu_1 \leqslant \mu_0$. It is easy to see that a language L of $\mathcal{B}$ satisfies the ultrafilter equation $\mu_0 = \mu_1$ if and only if L and L^c satisfy the ultrafilter inequality $\mu_0 \leqslant \mu_1$.

3. The operation $L \to LaA^*$

Let a be a letter of A and let u be a word of A^*. A word v is said to be an *a-prefix* of u if va is a prefix of u. Let $p_a(u)$ be the set of a-prefixes of u, that is,

$$p_a(u) = \{v \in A^* \mid va \text{ is a prefix of } u\}.$$

We view p_a as a *transduction* from A^* into itself. Following the notation introduced in [6], we set, for $L \subseteq A^*$,

$$p_a^-(L) = \{u \in A^* \mid p_a(u) \cap L \neq \emptyset\} \qquad p_a^+(L) = \{u \in A^* \mid p_a(u) \subseteq L\}$$

The link with the operation $L \to LaA^*$ comes from the following observation:

Proposition 3.1. *One has $p_a^-(L) = LaA^*$ and $p_a^+(L) = (L^c aA^*)^c$.*

We now extend the definition of the set of a-prefixes to βA^* and to $\widehat{A^*}$.

a-prefixes in βA^*. After explaining that βA^* is not a monoid, it may seem contradictory to define the set of a-prefixes of an ultrafilter. The key point is that since $\mathcal{C}(\beta A^*)$ is a compact space, any transduction of finite range from A^* to itself admits a unique continuous extension from βA^* to $\mathcal{C}(\beta A^*)$. This applies in particular to the map p_a. Thus if μ is an ultrafilter on A^*, we say that $\beta p_a(\mu)$ is the set of *a-prefixes* of μ and by abuse of language, we call *a-prefixes* of μ an element of $\beta p_a(\mu)$.

a-prefixes in $\widehat{A^*}$. It follows from [24, Theorem 4.1] that a map $f : A^* \to \mathcal{C}(\widehat{A^*})$ is uniformly continuous if and only if, for every regular language L, $f^{-1}(L)$ is a regular language[2]. Since $p_a^-(L) = LaA^*$ by Proposition 3.1, this condition is trivially satisfied by p_a and hence p_a extends uniquely to a (uniformly) continuous map $\widehat{p_a} : \widehat{A^*} \to \mathcal{C}(\widehat{A^*})$. The next proposition gives a direct definition of $\widehat{p_a}$.

[2]This is actually a variation on some general results of [17] and [33, Theorem 1]. See also [7] for a nice survey.

Proposition 3.2. *For each* $u \in \widehat{A^*}$,
$$\widehat{p_a}(u) = \{v \in \widehat{A^*} \mid \text{there exists } w \in \widehat{A^*} \text{ such that } u = vaw\}.$$

4. Inequalities for $\mathcal{L}_a$, and $\mathcal{L}_1$

Let a be a letter of A and let $\mathcal{L}$ be a lattice of languages. The aim of this section is to find out the ultrafilter inequalities satisfied by $\mathcal{L}_a$ and $\mathcal{L}_1$, and, when $\mathcal{L}$ is a lattice of regular languages, the profinite inequalities satisfied by these lattices.

4.1. **Ultrafilters inequalities.** The ultrafilter inequalities satisfied by $\mathcal{L}_a$ are described in the following theorem. A different description was given in [16].

Theorem 4.1. *Let* $\mu_0, \mu_1 \in \beta A^*$. *The following conditions are equivalent:*

(1) *The lattice* $\mathcal{L}_a$ *satisfies the inequality* $\mu_0 \leqslant \mu_1$,
(2) *The lattice* $\mathcal{L}$ *satisfies the inequality* $\mu_0 \leqslant \mu_1$ *and, for each* a*-prefix* γ_0 *of* μ_0, *there exists an* a*-prefix* γ_1 *of* μ_1 *such that* $\mathcal{L}$ *satisfies the inequality* $\gamma_0 \leqslant \gamma_1$.

Proof. In this proof, $\overline{S}$ denotes the closure in βA^* of a subset S of A^*.

Let $K_0 = \beta p_a(\mu_0)$ and $K_1 = \beta p_a(\mu_1)$. One can show that, for $i = 0, 1$, K_i is the unique compact subset of βA^* such that, for each $S \subseteq A^*$,
$$K_i \cap \overline{S} \neq \emptyset \text{ if and only if } SaA^* \in \mu_i. \tag{4.1}$$
(1) $\implies$ (2). If $\mu_0 \leqslant_{\mathcal{L}_a} \mu_1$, then, $\mu_0 \leqslant_{\mathcal{L}} \mu_1$ since $\mathcal{L} \subseteq \mathcal{L}_a$. The second part of (2) is trivially satisfied if $K_0 = \emptyset$. Thus we will now assume that K_0 is nonempty. Then since $\overline{A^*} = \beta A^*$, $K_0 \cap \overline{A^*}$ is also nonempty and (4.1) shows that $A^*aA^* \in \mu_0$. Since $\mu_0 \leqslant_{\mathcal{L}_a} \mu_1$, one also gets $A^*aA^* \in \mu_1$ and again by (4.1), K_1, which is equal to $K_1 \cap \overline{A^*}$, is nonempty.
Let $\gamma_0 \in K_0$. We claim that the set
$$\mathcal{S} = \{K_1 \cap \overline{L} \mid L \in \gamma_0 \cap \mathcal{L}\}$$
has the finite intersection property. Since, for $L_1, L_2 \in \mathcal{L}$, $\overline{L_1 \cap L_2} = \overline{L_1} \cap \overline{L_2}$, $\mathcal{S}$ is closed under finite intersection and it suffices to prove that $\mathcal{S}$ does not contain the empty set. But if $L \in \gamma_0 \cap \mathcal{L}$, then $\gamma_0 \in \overline{L}$ and thus $K_0 \cap \overline{L} \neq \emptyset$. It follows by (4.1) that $LaA^* \in \mu_0$. Now since $\mu_0 \leqslant_{\mathcal{L}_a} \mu_1$, one also gets $LaA^* \in \mu_1$ and again by (4.1), $K_1 \cap \overline{L} \neq \emptyset$. It follows that the elements of $\mathcal{S}$ are all nonempty, which proves the claim.

Since $\mathcal{C}(\beta A^*)$ is compact, the intersection of all elements of $\mathcal{S}$ is nonempty. Let γ_1 be an element of this intersection. Then since $A^* \in \gamma_0 \cap \mathcal{L}$, one gets in particular $\gamma_1 \in K_1 \cap \overline{A^*} = K_1$.

It just remains to show that $\gamma_0 \leqslant_{\mathcal{L}} \gamma_1$. Let $L \in \mathcal{L}$. If $L \in \gamma_0$, then $L \in \gamma_0 \cap \mathcal{L}$ and thus $K_1 \cap \overline{L} \in \mathcal{S}$ by definition of $\mathcal{S}$. Since γ_1 belongs to all elements of $\mathcal{S}$, we get in particular $\gamma_1 \in K_1 \cap \overline{L}$. It follows that $\gamma_1 \in \overline{L}$ and thus $L \in \gamma_1$ as required.

(2) $\implies$ (1). Suppose that (2) holds. Since $\mu_0 \leqslant_{\mathcal{L}} \mu_1$, it just remains to prove that, if $L \in \mathcal{L}$ and $LaA^* \in \mu_0$, then $LaA^* \in \mu_1$. Since $LaA^* \in \mu_0$, it follows from (4.1) that $K_0 \cap \overline{L}$ is nonempty. Let $\gamma_0 \in K_0 \cap \overline{L}$. Now, by (2) there exists $\gamma_1 \in K_1$ such that $\gamma_0 \leqslant_{\mathcal{L}} \gamma_1$. Since $\gamma_0 \in \overline{L}$, one gets $\gamma_1 \in \overline{L}$ and hence $\gamma_1 \in K_1 \cap \overline{L}$. It follows by (4.1) that $LaA^* \in \mu_1$ as required. □

Since $\mathcal{L}_1$ is the join of the lattices $\mathcal{L}_a$, for $a \in A$, one gets the following corollary.

Corollary 4.2. *Let $\mu_0, \mu_1 \in \beta A^*$. The following conditions are equivalent:*

(1) *The lattice $\mathcal{L}_1$ satisfies the inequality $\mu_0 \leqslant \mu_1$,*

(2) *The lattice $\mathcal{L}$ satisfies the inequality $\mu_0 \leqslant \mu_1$ and, for each letter a and for each a-prefix γ_0 of μ_0, there exists an a-prefix γ_1 of μ_1 such that $\mathcal{L}$ satisfies the inequality $\gamma_0 \leqslant \gamma_1$.*

4.2. **Profinite inequalities.** The counterparts of Theorem 4.1 and Corollary 4.2 for a lattice $\mathcal{L}$ of regular languages are stated below.

Theorem 4.3. *Let $u_0, u_1 \in \widehat{A^*}$. The following conditions are equivalent:*

(1) *The lattice $\mathcal{L}_a$ satisfies the inequality $u_0 \leqslant u_1$,*

(2) *The lattice $\mathcal{L}$ satisfies the inequality $u_0 \leqslant u_1$ and, for each a-prefix v_0 of u_0, there exists an a-prefix v_1 of u_1 such that $\mathcal{L}$ satisfies the inequality $v_0 \leqslant v_1$.*

Corollary 4.4. *Let $u_0, u_1 \in \widehat{A^*}$. The following conditions are equivalent:*

(1) *The lattice $\mathcal{L}_1$ satisfies the inequality $u_0 \leqslant u_1$,*

(2) *The lattice $\mathcal{L}$ satisfies the inequality $u_0 \leqslant u_1$ and, for each letter a and each a-prefix v_0 of u_0, there exists an a-prefix v_1 of u_1 such that $\mathcal{L}$ satisfies the inequality $v_0 \leqslant v_1$.*

5. A base of profinite inequalities for $\mathcal{L}_1$

In this section, we assume that $\mathcal{L}$ is a Boolean algebra of regular languages closed under quotients. In this case, $\mathcal{L}_1$ is a lattice of regular languages closed under quotients. It follows that the set of profinite inequalities satisfied by $\mathcal{L}_1$ is closed under translations: if $u_0 \leqslant u_1$ is satisfied by $\mathcal{L}_1$, then, for all $x, y \in \widehat{A^*}$, the inequality $xu_0y \leqslant xu_1y$ is also satisfied by $\mathcal{L}_1$.

A set E of profinite inequalities is a *base* for $\mathcal{L}_1$ if $\mathcal{L}_1$ is the smallest lattice of regular languages closed under quotients satisfying the inequalities of E. The aim of this section is to produce such a base of profinite inequalities.

As a Boolean algebra, $\mathcal{L}$ satisfies the profinite inequality $u_0 \leqslant u_1$ if and only if it satisfies $u_0 = u_1$. The profinite inequalities satisfied by $\mathcal{L}_1$ are described by Corollary 4.4: $u_0 \leqslant_{\mathcal{L}_1} u_1$ if and only if (u_0, u_1) satisfies the following conditions:

(C_1) $u_0 =_{\mathcal{L}} u_1$

(C_2) for each letter a and each a-prefix v_0 of u_0, there exists an a-prefix v_1 of u_1 such that $v_0 \leqslant_{\mathcal{L}} v_1$.

Let us consider the following subsets of $\widehat{A^*} \times \widehat{A^*}$:

$$E_1(\mathcal{L}) = \{(zx^2, zx) \mid x, z \in \widehat{A^*} \text{ and } z =_{\mathcal{L}} zx\}$$

$$E_2(\mathcal{L}) = \{(z, zx) \mid x, z \in \widehat{A^*} \text{ and } z =_{\mathcal{L}} zx\}$$

$$E_3(\mathcal{L}) = \{(zx_0x_1, zx_1x_0) \mid x_0, x_1, z \in \widehat{A^*} \text{ and } z =_{\mathcal{L}} zx_0 =_{\mathcal{L}} zx_1\}$$

$$E(\mathcal{L}) = E_1(\mathcal{L}) \cup E_2(\mathcal{L}) \cup E_3(\mathcal{L})$$

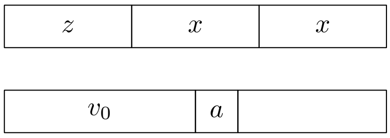
Let us first verify that the inequalities defined by $E(\mathcal{L})$ are all satisfied by $\mathcal{L}_1$.

Proposition 5.1. *Each profinite inequality* $u_0 \leqslant u_1$ *such that* $(u_0, u_1) \in E(\mathcal{L})$ *satisfies Conditions* (C$_1$) *and* (C$_2$) *and hence is satisfied by* $\mathcal{L}_1$*.*

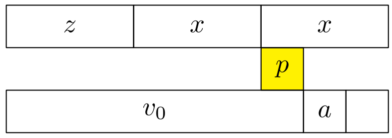
This is mainly a consequence of the equidivisibility of $\widehat{A^*}$.

Proof. (1) Suppose that $(u_0, u_1) \in E_1(\mathcal{L})$. Then there exist $x, z \in \widehat{A^*}$ such that $u_0 = zx^2$, $u_1 = zx$ and $z =_{\mathcal{L}} zx$. It follows $zx^2 =_{\mathcal{L}} zx$ and thus $u_0 =_{\mathcal{L}} u_1$.

Let a be a letter of A. If v_0a is a prefix of zx^2, then either v_0a is a prefix of zx or zx is a prefix of v_0.



First case.



Second case.

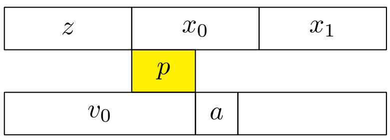
In the first case, it suffices to take $v_1 = v_0$ to get an a-prefix of zx such that $v_0 =_{\mathcal{L}} v_1$. In the second case, $v_0 = zxp$ for some p such that pa is a prefix of x. Let $v_1 = zp$. Then $v_1a = zpa$ is a prefix of zx and since $z =_{\mathcal{L}} zx$, $v_0 =_{\mathcal{L}} v_1$.

(2) Suppose that $(u_0, u_1) \in E_2(\mathcal{L})$. Then there exist $x, z \in \widehat{A^*}$ such that $u_0 = z$, $u_1 = zx$ and $z =_{\mathcal{L}} zx$. Thus the condition $u_0 =_{\mathcal{L}} u_1$ is trivially satisfied.

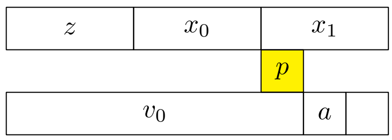
Let a be a letter of A. If v_0a is a prefix of z, then v_0a is also a prefix of zx and it suffices to take $v_1 = v_0$ to satisfy (C$_2$).

(3) Suppose that $(u_0, u_1) \in E_3(\mathcal{L})$. Then there exist $x_0, x_1, z \in \widehat{A^*}$ such that $u_0 = zx_0x_1$, $u_1 = zx_1x_0$ and $z =_{\mathcal{L}} zx_0 =_{\mathcal{L}} zx_1$. It follows $zx_0x_1 =_{\mathcal{L}} zx_1 =_{\mathcal{L}} z =_{\mathcal{L}} zx_0 =_{\mathcal{L}} zx_1x_0$ and thus $u_0 =_{\mathcal{L}} u_1$.

Let a be a letter of A. If v_0a is a prefix of zx_0x_1, then either v_0a is a prefix of zx_0 or zx_0 is a prefix of v_0.



First case.



Second case.

In the first case, $v_0 = zp$ for some p such that pa is a prefix of x_0. Let $v_1 = zx_1p$. Then $v_1a = zx_1pa$ is a prefix of zx_1x_0 and since $z =_{\mathcal{L}} zx_1$, $v_0 =_{\mathcal{L}} v_1$. In the second case, $v_0 = zx_0p$ for some p such that pa is a prefix of x_1. Let $v_1 = zp$. Then $v_1a = zpa$ is a prefix of zx_1x_0 and since $z =_{\mathcal{L}} zx_0$, $v_0 =_{\mathcal{L}} v_1$. $\square$

We can now state the main result of this section.

Theorem 5.2. *The inequalities of the form* $u_0 \leqslant u_1$ *such that* $(u_0, u_1) \in E(\mathcal{L})$ *form a base of profinite inequalities for* $\mathcal{L}_1$*. Alternatively, the equalities* $u_0 = u_1$ *such that* $(u_0, u_1) \in E_1(\mathcal{L}) \cup E_3(\mathcal{L})$ *and the inequalities* $u_0 \leqslant u_1$ *such that* $(u_0, u_1) \in E_2(\mathcal{L})$ *form another base for* $\mathcal{L}_1$*.*

We first show that the two sets of inequalities proposed in the statement define the same lattice of regular languages (Proposition 5.3).

Proposition 5.3. *The two sets of inequalities proposed in the statement of Theorem 5.2 define the same lattice of regular languages.*

Proof. Since the second set is larger than the first one, it suffices to show that the inequalities of the second set can be deduced from those of the first set.

The equation $zx = zx^2$ is equivalent to $zx^2 \leqslant zx$ and $zx \leqslant zx^2$. The inequality $zx^2 \leqslant zx$ is given by $E_1(\mathcal{L})$ and the other one is a consequence of $z \leqslant zx$, an inequality given by $E_2(\mathcal{L})$. Finally, the equation $zx_0x_1 = zx_1x_0$ follows from the inequalities $zx_0x_1 \leqslant zx_1x_0$ and $zx_1x_0 \leqslant zx_0x_1$, both given by $E_3(\mathcal{L})$. □

The end of the proof relies on a technical tool, the derived category of a relational morphims, and on Simon's theorem on free categories over a graph, or more precisely, its ordered version.

Derived category of a relational morphism. We refer to [31, 30, 34, 38] for more details on this topic. The ordered version was first introduced in [23].

Let M and N be finite ordered monoids and let $\tau : M \to N$ be a relational morphism. We define a category C_τ as follows: its objects are the elements of N and its arrows are of the form

$$n_0 \xrightarrow{(m,n)} n_1$$

where $n \in \tau(m)$ and $n_1 = n_0 n$. Composition of arrows is obtained by multiplying their labels:

$$n_0 \xrightarrow{(m,n)} n_1 \xrightarrow{(m',n')} n_2 \quad = \quad n_0 \xrightarrow{(mm',nn')} n_2$$

The identity at the object n, denoted by 1_n, is the arrow

$$n \circlearrowleft (1,1)$$

Two arrows are *coterminal* if they have same origin and same end. Given two coterminal arrows from n_0 to n_1, we write

$$n_0 \xrightarrow{(m,n)} n_1 \quad \preccurlyeq \quad n_0 \xrightarrow{(m',n')} n_1$$

if, for every $m_0 \in \tau^{-1}(n_0)$, one has $m_0 m \leqslant m_0 m'$. This defines a preorder on the set of arrows of C_τ which is compatible with the product in C_τ.

Let $\sim$ be the congruence associated with $\preccurlyeq$. Thus

$$n_0 \xrightarrow{(m,n)} n_1 \quad \sim \quad n_0 \xrightarrow{(m',n')} n_1$$

if, for all $m_0 \in \tau^{-1}(n_0)$, one has $m_0 m = m_0 m'$. The *derived category* of τ, denoted D_τ, is the quotient of C_τ by $\sim$. The *ordered derived category* of τ is the derived category equipped with the order induced by the preorder $\preccurlyeq$ in C_τ.

A result on finite ordered categories Let C be a category and let p be an arrow of the free category over C (that is, a path in the directed graph

C). We denote by $c(p)$ the *content* of p, that is, the set of arrows occurring in p. The following statement [23, Prop. 4.2] is the counterpart for ordered categories of a celebrated theorem of I. Simon on categories.

Proposition 5.4. *Let C be a finite ordered category. The following conditions are equivalent:*

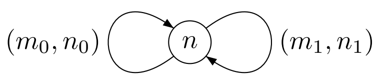
1. *The local monoids of C satisfy the monoid identities $x^2 = x$, $xy = yx$ and $1 \leqslant x$.*
2. *C satisfies the inequality $x \leqslant y$ for any coterminal arrows x and y of the free category over C such that $c(x) \subseteq c(y)$.*

Sketch of the proof of Theorem 5.2. We first give a slightly more precise result when $\mathcal{L}$ is a finite Boolean algebra (Proposition 5.5). The general case follows by a compactness argument (omitted).

Proposition 5.5. *Let $\mathcal{L}$ be a finite Boolean algebra of regular languages of A^* closed under quotients. Then $\mathcal{L}_1$ is defined by the set of inequalities $u_0 \leqslant u_1$ such that $(u_0, u_1) \in E(\mathcal{L}) \cap (A^* \times A^*)$.*

Proof. Let $E'(\mathcal{L}) = E(\mathcal{L}) \cap (A^* \times A^*)$. Let L be a regular language of A^* satisfying the inequalities defined by $E'(\mathcal{L})$. Let us show that L satisfies all the inequalities $u_0 \leqslant u_1$ such that Conditions (C$_1$) and (C$_2$) are satisfied.

Let $\mathcal{L} = \{L_1, \cdots, L_n\}$. For $1 \leqslant i \leqslant n$, let $\eta_i : A^* \to M_i$ be the syntactic morphism of L_i and let $\eta : A^* \to M_1 \times \cdots \times M_n$ be the diagonal morphism defined by $\eta(u) = (\eta_1(u), \ldots, \eta_n(u))$. Let $\mu : A^* \to M$ be the syntactic morphism of L. Finally, let $N = \eta(A^*)$ and let τ be the relational morphism $\eta \circ \mu^{-1} : M \to N$. We claim that every local monoid of D_τ satisfies the inequalities $x^2 = x$, $xy = yx$ and $x \leqslant 1$. Let $n \in N$ and consider two loops $r = (n, (m_0, n_0), n)$ and $s = (n, (m_1, n_1), n)$ around n.



Then $n_0 \in \tau(m_0)$, $n_1 \in \tau(m_1)$ and $n = nn_0 = nn_1$. Let z, x_0 and x_1 be words of A^* such that $\eta(z) = n$, $\eta(x_0) = n_0$, $\mu(x_0) = m_0$, $\eta(x_1) = n_1$ and $\mu(x_1) = m_1$. Then $\eta(zx_0) = \eta(zx_1) = \eta(z)$, which means that $zx_0 =_{\mathcal{L}} zx_1 =_{\mathcal{L}} z$. Since L satisfies the inequalities defined by $E'(\mathcal{L})$, one has $\mu(z) \leqslant \mu(zx_0) = \mu(zx_0^2)$ and $\mu(zx_0x_1) = \mu(zx_1x_0)$. Consequently, $1_n \preccurlyeq r$, $r^2 \sim r$ and $rs \sim sr$, which proves the claim. We can now apply Proposition 5.4 to the ordered derived category of τ. Let $(u_0, u_1) \in A^* \times A^*$ satisfying Conditions (C$_1$) and (C$_2$).

With each $w = a_1 \cdots a_n \in A^*$, where $a_1, \ldots, a_n \in A$, we associate a path $p(w)$ in D_τ as follows:

$$1 \xrightarrow{[(1,a_1)]_\sim} \eta(a_1) \xrightarrow{[(\eta(a_1),a_2)]_\sim} \eta(a_1a_2) \cdots \xrightarrow{[(\eta(a_1\cdots a_{n-1}),a_n)]_\sim} \eta(w).$$

Now $u_0 =_{\mathcal{L}} u_1$ if and only if $\eta(u_0) = \eta(u_1)$, Condition (C$_1$) is equivalent to $p(u_0)$ and $p(u_1)$ be coterminal, and (C$_2$) is equivalent to $c(p(u_0)) \subseteq c(p(u_1))$. It follows now from Proposition 5.4 that $p(u_0) \preccurlyeq p(u_1)$. This means that $\mu(zu_0) \leqslant \mu(zu_1)$ for any $z \in \eta^{-1}(1)$ and hence $\mu(u_0) \leqslant \mu(u_1)$. Thus L satisfies $u_0 \leqslant u_1$. □

Conclusion. The programme would now be to extend the known results on operations on regular languages to the framework of lattices. So far, only the polynomial closure was understood [8]. This paper solves the case of one-step products, but a challenging problem would be to extend the results of [26, 35] to lattices.

Classes of languages $\rightarrow$ $\downarrow$ **Operations**	(Positive) Varieties	Lattices of regular languages
Lattice generated by one step products	[28, Th. 4.6]	This paper
Polynomial closure	[27]	[8]
Closure under product and Boolean operations	[35, 26]	Open

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Mário J. J. Branco: CEMAT-Ciências and Dep. Matemática da Faculdade de Ciências da Universidade de Lisboa, Campo Grande, 1749-016 Lisboa, Portugal
Email address: mjbranco@fc.ul.pt

Jean-Éric Pin: IRIF, Université Paris Denis Diderot et CNRS - Case 7014 - F-75205 Paris Cedex 13 FRANCE.
Email address: Jean-Eric.Pin@irif.fr