REMARKS ON A CONJECTURE ON CERTAIN INTEGER SEQUENCES

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Abstract. The periodicity of sequences of integers $(a_n)_{n\in\mathbb{Z}}$ satisfying the inequalities

$$0 \le a_{n-1} + \lambda a_n + a_{n+1} < 1 \ (n \in \mathbb{Z})$$

is studied for real λ with $|\lambda| < 2$. Periodicity is proved in case λ is the golden ratio; for other values of λ statements on possible period lengths are given. Further interesting results on the morphology of periods are illustrated.

The problem is connected to the investigation of shift radix systems and of Salem numbers.

1. Introduction

In this note we will analyze the following conjecture (see [1]):

Conjecture 1.1. Let $\lambda \in \mathbb{R}$ and assume that the sequence of integers $(a_n)_{n\in\mathbb{Z}}$ satisfies the inequalities

$$0 \le a_{n-1} + \lambda a_n + a_{n+1} < 1 \ (n \in \mathbb{Z}). \tag{1.1}$$

If $|\lambda| < 2$ then $(a_n)_{n\in\mathbb{Z}}$ is periodic.

The conjecture is supported by extensive computer experiments and by some theorems, which we will collect below. It is trivially true for $\lambda = -1, 0, 1$.

The conjecture seems to be interesting by itself, but there are also connections to other areas. Firstly, let us recall the definition of a shift radix system. To a vector $\mathbf{r} \in \mathbb{R}^d$ we associate the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: If $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ then let[1]

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra} = r_1a_1 + \cdots + r_da_d$, i.e. the usual inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. Then the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \longrightarrow \mathbb{Z}^d$ is called a shift radix system if for every $\mathbf{a} \in \mathbb{Z}^d$ the orbit $(\tau_{\mathbf{r}}^k(\mathbf{a}))_{k\in\mathbb{N}}$ ends up in the zero cycle. In general, it is a hard problem to decide which $\mathbf{r} \in \mathbb{R}^d$ define shift radix systems. Clearly, it is considerably easier to investigate the set $\mathcal{D}_d$ of those $\mathbf{r} \in \mathbb{R}^d$ which yield ultimately periodic orbits for each $\mathbf{a} \in \mathbb{Z}^d$. But even in dimension $d = 2$ this easier problem is not completely settled. It is believed [1] that the line segment

$$\{(1, \lambda) \in \mathbb{R}^2 || \lambda| < 2\}$$

belongs to $\mathcal{D}_2$. Going back to the definitions we realize that we are exactly left with the problem stated in Conjecture 1.1.

Secondly, there is a connection to a famous open question on Salem numbers, which are algebraic integers where all conjugates have absolute value ≤ 1, with at least one conjugate on the unit circle. Bertrand and K. Schmidt proved independently that the set Per(β) of all numbers with ultimately periodic β-expansion is exactly $\mathbb{Q}(\beta)$ if β is a Pisot number (an algebraic integer with all conjugates lying strictly inside the unit circle) and that Per(β) = $\mathbb{Q}(\beta)$ implies that β is a Pisot number or

Date: August 12, 2005.

2000 *Mathematics Subject Classification.* 11B37, 11B39 .

Key words and phrases. integer sequences, periodicity.

The first author was supported by the Japan Society for the Promotion of Science, Grants-in Aid for fundamental research 14540015, 2002–2005.

The third author was supported partially by the Hungarian National Foundation for Scientific Research Grant Nos. T42985 and T38225.

The fourth author was supported by the Austrian Science Fund FWF, grant S8302-MAT.

[1] $\{x\}$ denotes the fractional part of x and $\lfloor x \rfloor$ denotes its integer part.

a Salem number, but we still do not know for any Salem number if this property is true or false. In Hollander [4] and Rigo and Steiner [6] (in a more general context), an equivalent formulation of this problem was given implicitly: Let β be a root of $x^d - b_1 x^{d-1} - \cdots - b_d$ with $b_j \in \mathbb{Z}$, $r_j = \frac{b_{d-j+2}}{\beta} + \cdots + \frac{b_d}{\beta^{j-1}}$ (with $r_1 = 0$) and extend the definition of $\tau_{\mathbf{r}}$ to $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Q}^d$ by

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{ra} + \{b_d a_1 + \cdots + b_1 a_d\} \rfloor + \{b_d a_1 + \cdots + b_1 a_d\}).$$

Then we have $\mathrm{Per}(\beta) = \mathbb{Q}(\beta)$ if and only if $(\tau_{\mathbf{r}}^k(\mathbf{a}))_{k\in\mathbb{N}}$ is ultimately periodic for all $\mathbf{a} \in \mathbb{Q}^d$. Note that, for $\mathbf{a} \in \frac{1}{q}\mathbb{Z}^d$, the iterates $\tau_{\mathbf{r}}^k(\mathbf{a})$ are in $\frac{1}{q}\mathbb{Z}^d$ as well and that a_1 can be replaced by $\{b_d a_1 + \cdots + b_1 a_d\}$, hence we have a shift radix system in $\frac{1}{q}(\mathbb{Z}_q \times \mathbb{Z}^{d-1})$, which can probably be treated similar to a shift radix system in $\mathbb{Z}^{d-1}$.

Conjecture 1.1 does not give answers to this problem because the degree of Salem numbers is at least 4 which implies that the corresponding shift radix system has at least degree 3, but the problems are similar: The eigenvalues of the shift radix system corresponding to a Salem number β are the conjugates of β and lie therefore inside the unit circle (with at least one lying on it), the eigenvalues of the shift radix system (1.1) lie on the unit circle if $|\lambda| < 2$.

Boyd [2] provided some heuristics predicting that $\mathrm{Per}(\beta) = \mathbb{Q}(\beta)$ holds for Salem numbers of degree 4 and 6, but not for Salem numbers of higher degree. Since Salem numbers are reciprocal, $1/\beta$ is a conjugate of β and all other conjugates lie on the unit circle. Therefore we have one contracting direction of the shift radix system and a $(d-2)$-dimensional hyperplane corresponding to the other eigenvalues. Therefore $||\tau_{\mathbf{r}}^k(\mathbf{a})|| = \mathcal{O}\left(k^\delta\right)$ for some $\delta < 1/(d-2)$ implies that $(\tau_{\mathbf{r}}^k(\mathbf{a}))_{k\in\mathbb{N}}$ is ultimately periodic (cf. Proposition 2.4). Boyd assumed that the $\tau_{\mathbf{r}}^k(\mathbf{a})$ are randomly distributed and used the birthday paradox to show that $\delta < 2/(d-2)$ is sufficient if this assumption holds. By a random walk argument, he gave a justification for $\delta = 1/2$.

2. General results

We start with a property satisfied for all λ. A sequence of integers $(a_n)_{n\in\mathbb{Z}}$ is called an orbit belonging to λ (or λ-orbit for short), if (1.1) holds for all $n \in \mathbb{Z}$. To study orbits it is more convenient to consider their consecutive terms because any two of them generate the whole orbit. We say that $(x, y) \in \mathbb{Z}^2$ lies on an orbit if x and y are consecutive terms of the orbit. Fix λ and call $(a_1, a_2), (b_1, b_2) \in \mathbb{Z}^2$ equivalent if they lie on the same orbit. It is clear that this is an equivalence relation on $\mathbb{Z}^2$. Hence any pair $(a_1, a_2) \in \mathbb{Z}^2$ belongs to an orbit and two different orbits belonging to λ have no common consecutive elements.

Lemma 2.1. *Let $(a_n)_{n\in\mathbb{Z}}$ be a λ-orbit , assume $a_k = a_{k+1}$ for some $k \in \mathbb{Z}$ and $a_{k+\ell+1} = a_{k+\ell+2}$ for some $\ell \in \mathbb{N}$. Then $(a_n)_{n\in\mathbb{Z}}$ is periodic with period length $2\ell + 2$.*

Proof. Without loss of generality we may assume $k = 1$. By induction on $|n|$ we show $a_{n+2} = a_{1-n}$ for all $n \in \mathbb{Z}$. Therefore, if $a_{k+\ell+1} = a_{k+\ell+2}$ then $(a_{-\ell}, a_{-\ell+1}) = (a_{2+\ell}, a_{3+\ell})$. □

Theorem 2.2. *There exist infinitely many orbits belonging to λ.*

Proof. Consider an orbit with equal starting values, i.e. assume $a_1 = a_2$. Now if there exists an $i \geq 0$ such that $a_{2+i} = a_{3+i}$ then by Lemma 2.1 $(a_n)_{n\in\mathbb{Z}}$ becomes periodic. Hence $(a_n)_{n\in\mathbb{Z}}$ can include at most two diagonal pairs. As there exist infinitely many diagonal pairs the proof is done. □

The following examples shed some light on the growth of possible period lengths of orbits.

Example 2.3. Let $0 < \lambda < 1$ and q be a positive integer.

(i) The λ-orbit with initial values $0, q$ where $q < \frac{1}{1-\lambda}$ has minimal period length $6q + 1$ (see [1]).

(ii) For $\lambda < \frac{1}{q}$ the λ-orbit with initial values $0, q$ has minimal period length $4q+3$. This can easily be seen by calculating the first eight elements of the orbit, i.e. $0, q, 0, -q, 1, q, -1, -q+1$, and then deducing

$$a_{4k+1} = k, a_{4k+2} = q - k + 1, a_{4k+3} = -k, a_{4k+4} = -q + k \quad (1 \leq k \leq q)$$

by induction. We conclude $a_{4q+4} = 0, a_{4q+5} = q$.

In the sequel we assume $|\lambda| < 2$. Then there exists a unique $\alpha \in [0, \pi)$ such that $\lambda = 2\cos\alpha$. Setting $\omega = \cos\alpha + i\sin\alpha$ we get $\lambda = \omega + \bar{\omega}$, where $i = \sqrt{-1}$ and $\bar{\omega}$ denotes the conjugate of the complex number ω.

To a λ-orbit $(a_n)_{n\in\mathbb{Z}}$ we associate some other sequences which make our investigation more tractable. First set $b_{n-1} = a_{n-1} + \lambda a_n + a_{n+1}$ for $n \in \mathbb{Z}$. Then (1.1) implies $0 \le b_n < 1$, and by the definition of ω we have

$$b_{n-1} = a_{n-1} + \omega a_n + \bar{\omega}(a_n + \omega a_{n+1}) \ \ (n \in \mathbb{Z}).$$

Hence it is convenient to introduce the companion sequence $c_n = a_n + \omega a_{n+1}$ $(n \in \mathbb{Z})$. Rewriting this relation we obtain

$$b_{n-1} = c_{n-1} + \bar{\omega} c_n. \tag{2.1}$$

For the sequence of real numbers defined by $\alpha_{n+1} = -\alpha_{n-1} - \lambda\alpha_n$, the points (α_n, α_{n+1}) lie on the ellipse

$$r^2 = \alpha_n^2 + \lambda\alpha_n\alpha_{n+1} + \alpha_{n+1}^2 = (2+\lambda)\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right)^2 + (2-\lambda)\left(\frac{\alpha_n - \alpha_{n+1}}{2}\right)^2.$$

for some $r \ge 0$. For $a_{n+1} = -a_{n-1} - \lambda a_n + \{\lambda a_n\}$, the situation is slightly different. If we consider the companion sequence

$$r_n^2 = a_n^2 + \lambda a_n a_{n+1} + a_{n+1}^2 = (2+\lambda)\left(\frac{a_n + a_{n+1}}{2}\right)^2 + (2-\lambda)\left(\frac{a_n - a_{n+1}}{2}\right)^2,$$

we obtain

$$\begin{aligned} r_n^2 &= a_n^2 + \lambda a_n a_{n-1} + a_{n-1}^2 + \{\lambda a_n\}(-2a_{n-1} - \lambda a_n + \{\lambda a_n\}) \\ &= r_{n-1}^2 - \{\lambda a_n\}(2a_{n-1} + \lfloor \lambda a_n \rfloor) \\ &= r_{n-1}^2 - \{2a_{n-1} + \lambda a_n\}\lfloor 2a_{n-1} + \lambda a_n \rfloor. \end{aligned}$$

Proposition 2.4. *If*

$$\limsup_{n\to\infty} \frac{r_n^2}{n} < \frac{\sqrt{4-\lambda^2}}{\pi},$$

then $(a_n)_{n\in\mathbb{Z}}$ *is periodic.*

Proof. The area of the ellipse $a_n^2 + \lambda a_n a_{n+1} + a_{n+1}^2 = r_n^2$ is $r_n^2\pi/\sqrt{4-\lambda^2}$. The ellipse contains therefore $r_n^2\pi/\sqrt{4-\lambda^2} + \mathcal{O}(r_n)$ integer points. If this number grows slower than n, then two points (a_j, a_{j+1}) and (a_k, a_{k+1}) with $j \ne k$ are equal and the sequence is periodic. □

Obviousy every bounded λ-orbit is periodic. In the sequel we prove that already one-side boundedness is enough to prove periodicity.

Proposition 2.5. *If a λ-orbit is lower or upper bounded (one-side bounded) then it is periodic.*

To prove this statement we need two lemmata.

Lemma 2.6. *If there exists some $x \in \mathbb{Z}$ which appears infinitely often as a member of a one-side bounded λ-orbit (a_n) then (a_n) is periodic.*

Proof. Let J_x denote the (infinite) set of those indices j with $a_j = x$. Then

$$-\lambda x \le a_{j-1} + a_{j+1} < -\lambda x + 1$$

holds by (1.1) for all $j \in J_x$. Thus

$$-a_{j-1} - \lambda x \le a_{j+1} < -a_{j-1} - \lambda x + 1. \tag{2.2}$$

Assume that a_{j-1} is unbounded for $j \in J_x$. As (a_n) is one-side bounded $a_n < K$ $(a_n > -K)$ holds for all $n \in \mathbb{Z}$ with a constant K. Thus there exists $j \in J_x$ with $a_{j-1} \le -K - \lambda x$ $(a_{j-1} \ge K - \lambda x + 1)$. Then $a_{j+1} \ge K$ $(a_{j+1} < -K)$ holds by (2.2), which contradicts the assumption on (a_n).

Therefore a_{j-1} is bounded for all $j \in J_x$ and there exists $y \in \mathbb{Z}$ and $j_1, j_2 \in J_x, j_1 \neq j_2$ with $a_{j_1-1} = a_{j_2-1} = y$. Then $(a_{j_1-1}, a_{j_1}) = (a_{j_2-1}, a_{j_2}) = (y, x)$ and (a_n) is periodic by Lemma 2.1. □

Lemma 2.7. *Every unbounded λ-orbit has infinitely many positive and infinitely many negative terms.*

Proof. A collection of consecutive positive (negative) terms of (a_n) will be called a positive (negative) run. We prove that a run is always finite. This implies the statement immediately. We distinguish three cases according to the size of λ.

Case I, $\lambda \geq 0$. By (1.1) the length of a run is at most two.

Case II, $-1 \leq \lambda < 0$. Summing up three consecutive inequalities (1.1) we obtain

$$0 \leq a_{n-2} + (1+\lambda)a_{n-1} + (2+\lambda)a_n + (1+\lambda)a_{n+1} + a_{n+2} < 3.$$

Thus the length of a run is at most four.

Case III, $-2 < \lambda < -1$. This is the most complicated case. Consider first a positive run. We prove that it has a local maximum. Indeed, assume that the run starting with a_0 is increasing. Let ℓ be so large that $(\ell-2)(2+\lambda)+1+\lambda \geq 0$. We may assume that $a_0 \geq \ell$. By Lemma 2.1 two consecutive terms of (a_n) can be identical only once, thus we can achieve $a_0 \geq \ell$ by omitting from consideration at most ℓ terms.

Summing up ℓ consecutive inequalities (1.1) and using that the run is increasing we obtain

$$\begin{aligned} \ell &> a_0 + (1+\lambda)a_1 + (2+\lambda)\sum_{j=2}^{\ell-1} a_j + (1+\lambda)a_\ell + a_{\ell+1} \\ &\geq a_0 + (1+\lambda)a_1 + (2+\lambda)(\ell-2)a_2 \\ &\geq a_0 \geq \ell, \end{aligned}$$

which is a contradiction. Thus an increasing part of a run has length at most $2\ell+1$, which proves the claim.

Now let a_0 be a local maximum of our run. Only a_1 or a_{-1}, but not both can be equal to a_0, thus we may assume $a_0 > a_1$. Assume that $a_k > a_{k+1}$ for some $k \geq 0$. Dividing (1.1) by a_{k+1} we obtain

$$0 \leq \frac{a_k}{a_{k+1}} + \lambda + \frac{a_{k+2}}{a_{k+1}} < \frac{1}{a_{k+1}},$$

which implies

$$\frac{a_{k+2}}{a_{k+1}} < -\frac{a_k - 1}{a_{k+1}} - \lambda \leq -1 - \lambda < 1,$$

i.e. $a_{k+1} > a_{k+2}$. Thus from the maximum on the run is strictly decreasing. As the definition of (a_n) is symmetrical, this holds for both directions. Summing up the length of a positive run is at most $4\ell+3$.

Negative runs can be treated similarly. □

Proof of Proposition 2.5. If the λ-orbit (a_n) is bounded then we have nothing to do. Assume that it is one-side bounded, but unbounded. Then by Lemma 2.7 it has infinitely many positive and infinitely many negative terms. As (a_n) is one-side bounded there exists an integer which appears infinitely often in (a_n). By Lemma 2.6 (a_n) is bounded, hence periodic. □

3. General results on the periods

Although the following lemma is a simple consequence of the definitions it plays an important role in the analysis of periods.

Lemma 3.1. *Let $m \geq 1$ be an integer, then*

$$\sum_{j=1}^{m} b_{j-1}(-\omega)^{j-1} = c_0 - (-\omega)^m c_m. \tag{3.1}$$

Proof. Relation (3.1) is true for $m = 1$. Assume that it holds for m. Then using (3.1) and (1.1) we get

$$\begin{aligned}\sum_{j=1}^{m+1} b_{j-1}(-\omega)^{j-1} &= c_0 - (-\omega)^m c_m + b_m(-\omega)^m \\ &= c_0 - (-\omega)^m c_m + (-\omega)^m c_m + (-\omega)^m \omega c_{m+1} \\ &= c_0 - (-\omega)^{m+1} c_{m+1},\end{aligned}$$

and the lemma is proved. □

The meaning of the next theorem is that the minimal period length of a periodic orbit grows generally with the initial terms. More precisely we prove:

Theorem 3.2. *Assume that the orbit* $(a_n)_{n\in\mathbb{Z}}$ *belongs to* $\lambda = 2\Re\omega$ *and that* $(a_n)_{n\in\mathbb{Z}}$ *is periodic with minimal period length* p. *If*

(i) ω *is not a root of unity or*
(ii) *if* k *is the smallest positive integer with* $\omega^k = -1$ *and* k *does not divide* p *or*
(iii) *if* k *is the smallest positive integer with* $\omega^k = -1$, k *divides* p, k *is even and* p/k *is odd*

then $|a_0| + |a_1|$ *is bounded by a constant which depends only on* p *and on the argument of* ω.

Proof. As p is a period length of $(a_n)_{n\in\mathbb{Z}}$ we have $c_p = a_p + \bar{\omega} a_{p+1} = a_0 + \bar{\omega} a_1 = c_0$. Using this and (3.1) we get

$$\sum_{j=1}^{p} b_{j-1}(-\omega)^{j-1} = c_0(1 - (-\omega)^p). \tag{3.2}$$

The absolute value of the sum on the left hand side is at most p. It remains to prove that under one of the conditions (i)-(iii) $|1 - (-\omega)^p|$ is bounded below by a constant depending only on p and on the argument of ω. Note that by an elementary argument the boundedness of $|c_0|$ implies that $|a_0| + |a_1|$ is bounded.

The statement is obviously true when ω is not a root of unity.

In the cases (ii) and (iii) write $p = kt + r$, with $0 \le r < k$. Then

$$1 - (-\omega)^p = 1 + (-1)^{p+t+1}\omega^r.$$

If k does not divide p, then $r \neq 0$, i.e. $1 + (-1)^{p+t+1}\omega^r \neq 0$. Similarly, if k divides p, but k is even, then p is even and as $p/k = t$ is odd then $p + t + 1$ is even, i.e. $1 + (-1)^{p+t+1}\omega^r = 2$. □

The next corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let* $\omega \in \mathbb{C}$ *be not a root of unity and assume that Conjecture 1.1 holds for* $\lambda = 2\Re\omega$. *Then there exist periodic orbits belonging to* λ *with arbitrary long minimal period lengths.*

4. Morphology of the periods

According to our computation experience the graph of an orbit can have three different shapes. Typical examples are displayed in Figures 1-3. In Case I the corresponding ω is not a root of unity and the graph resembles an ellipse. In the other cases ω is an eighth root of unity. The difference is in the length of period. In Case II this number is not divisible by eight. The graph has only one component. Finally, in Case III the period length is divisible by eight and the graph consists of eight "connected" components. We show that this happens always.

Assume first that ω is a root of unity, i.e. if α is a rational multiple of π. Then we can associate a fourth, bounded sequence to $(a_n)_{n\in\mathbb{Z}}$.

Lemma 4.1. *Assume that* $\omega \neq -1$, *but* $\omega^k = -1$ *for some integer* $k > 1$ *and put*

$$d_n = \begin{cases} a_n - a_{n+k} & \text{, if } k \text{ is odd,} \\ a_n + a_{n+k} & \text{, if } k \text{ is even.} \end{cases}$$

Then $|d_n|$ *is bounded by a constant depending only on* k.

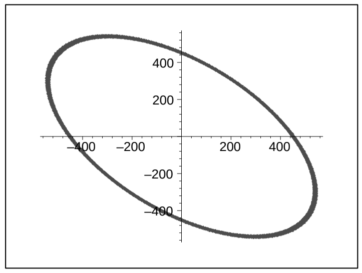

FIGURE 1. Case I. $\lambda = 1.1, a_1 = 25, a_2 = 462$, period length 7555

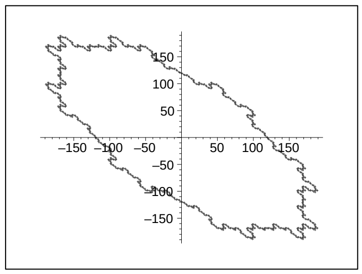

FIGURE 2. Case II. $\lambda = \sqrt{2}, a_1 = 169, a_2 = -169$, period length 1461

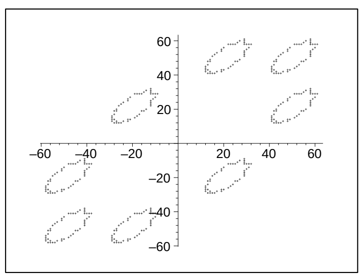

FIGURE 3. Case III. $\lambda = -\sqrt{2}, a_1 = 58, a_2 = 58$, period length 392

Proof. As we can start the orbit by any of its members it is enough to prove the lemma for $n = 1$. By (3.1) we get

$$\begin{aligned}\sum_{j=1}^{k} b_{j-1}(-\omega)^{j-1} &= c_0 - (-\omega)^k c_k \\ &= c_0 + (-1)^{k+2} c_k \\ &= a_0 + (-1)^k a_k + \bar{\omega}(a_1 + (-1)^k a_{k+1}) \\ &= d_0 + \bar{\omega} d_1.\end{aligned}$$

We have $\omega = \cos\frac{\ell\pi}{k} + i \sin\frac{\ell\pi}{k}$ for some odd $\ell \neq k$, $0 < \ell < 2k$. Comparing the imaginary parts of the relation we get

$$\sum_{j=1}^{k} b_{j-1}(-1)^j \sin\frac{(j-1)\ell\pi}{k} = d_1 \sin\frac{\ell\pi}{k}.$$

As the absolute value of the left hand side is less than k and $|\sin\frac{\ell\pi}{k}| \geq |\sin\frac{\pi}{k}|$, which depends only on k, the lemma is proved. □

To analyze the shape of (periodic) orbits it is more convenient to deal with the difference rather than the sum of members of $(a_n)_{n\in\mathbb{Z}}$. If k is even then

$$|a_n - a_{n+2k}| \le |a_n + a_{n+k}| + |a_{n+k} + a_{n+2k}| \le d_n + d_{n+k},$$

hence $|a_n - a_{n+2k}|$ is bounded, too. Thus for $\lambda = 2\Re\omega$ with $\omega^k = -1$ the following constant is well defined:

$$d_\lambda(k) = \begin{cases} \max\{|a_n - a_{n+k}| \,|\, (a_n)_{n\in\mathbb{Z}} \ \lambda\text{-orbit}\} & \text{, if } k \text{ is odd,} \\ 2\max\{|a_n - a_{n+2k}| \,|\, (a_n)_{n\in\mathbb{Z}} \ \lambda\text{-orbit}\} & \text{, if } k \text{ is even.} \end{cases}$$

Let $d \in \mathbb{N}$. A subset $S \subseteq \mathbb{Z}^2$ is called *d-connected* if $|S| = 1$ or if for any $\underline{a} \in S$ there exists $\underline{b} \in S, \underline{a} \ne \underline{b}$ with $\| \underline{a} - \underline{b} \|_\infty \le d$.

Theorem 4.2. *Let $\omega^k = -1$, $\lambda = 2\Re\omega$ and $d = d_\lambda(k)$ be defined as above. Let p be the minimal period length of the λ-orbit $(a_n)_{n\in\mathbb{Z}}$, and set $S = \{(a_n, a_{n+1}) \,|\, n \in \mathbb{Z}\}$.*

(i) *If $\gcd(k,p) = 1$, then S is d-connected.*

(ii) *If k is odd and divides p or if k is even and $2k$ divides p then S is the union of k or $2k$ d-connected subsets $S_0, \dots, S_{t-1}, t = k$ or $t = 2k$. Moreover, if $\max\{|a_n| \,|\, n \in \mathbb{Z}\}$ is large enough then $\| \underline{x}_u - \underline{x}_v \|_\infty > d$ for all $\underline{x}_u \in S_u, \underline{x}_v \in S_v, u \ne v$.*

Proof. Notice that $S = \{(a_n, a_{n+1}) \,|\, 0 \le n < p\}$. Assume first $\gcd(k,p) = 1$. Then the set $\{k\ell \,|\, 0 \le \ell < p\}$ is a complete residue system modulo p, hence

$$S = \{(a_{k\ell}, a_{k\ell+1}) \,|\, 0 \le \ell < p\}.$$

We also have

$$\| (a_{k\ell}, a_{k\ell+1}) - (a_{k(\ell+1)}, a_{k(\ell+1)+1}) \|_\infty \le d$$

by the definition of d. Thus S is indeed d-connected.

Now we turn to the proof of the second statement. Set $t = k$, if k is odd and divides p and $t = 2k$, if k is even and $2k$ divides p. Let

$$S_j = \{(a_n, a_{n+1}) \,|\, n \in \mathbb{Z}, n \equiv j \pmod t\} \ (j = 0, \dots, t-1).$$

We obviously have $\cup_{j=0}^{t-1} = S$ and claim $S_u \cap S_v = \emptyset$ whenever $u \ne v$. Indeed, if there would exist $(x,y) \in S_u \cap S_v$ then $(x,y) = (a_{u+t\ell}, a_{u+t\ell+1}) = (a_{v+th}, a_{u+th+1})$ would hold with some $\ell, h \in \mathbb{Z}$ such that $0 \le u + t\ell, v + th < p$. This implies $u + t\ell = v + th$ because two consecutive terms define $(a_n)_{n\in\mathbb{Z}}$ uniquely. Hence t divides $|u - v|$, which implies $u = v$.

By the definition of d it is clear that the sets $S_j, j = 0, \dots t-1$ are d-connected. It remains only to prove that the distance of S_u and S_v is large, provided that the largest term of $(a_n)_{n\in\mathbb{Z}}$ is large enough. Let $K = \max\{|a_n| \,|\, 0 \le n < p\}$. We may assume without loss of generality that $K = |a_0|$. Let m denote an index such that $|a_{mt}| = \min\{|a_{jt}| \,|\, 0 \le j < p/t\}$. If $j \le \lfloor p/(2t) \rfloor$then

$$|a_0 - a_{jt}| = |a_0 - a_t + a_t - a_{2t} + \cdots - a_{(j-1)t} + a_{(j-1)t} - a_{jt}| \le jd \le \frac{pd}{2t},$$

which implies

$$|a_{jt}| \ge K - \frac{pd}{2t}.$$

Otherwise, if $j > \lfloor p/(2t) \rfloor$ then using $a_p = a_0$ we get

$$|a_0 - a_{jt}| = |a_p - a_{p-t} + a_{p-t} - a_{p-2t} + \cdots - a_{(j+1)t} + a_{(j+1)t} - a_{jt}| \le \left(\frac{p}{t} - (j+1)\right) d \le \frac{pd}{2t}.$$

Thus we have

$$|a_{jt}| \ge K/2 \ \ (0 \le j < p/t), \tag{4.1}$$

provided K is large enough.

For $j \in \mathbb{Z}$ set $c_j = a_j + \bar{\omega} a_{j+1}$. By Lemma 3.1 the set S_j is close to the set $(-\omega)^{-j} S_0$. In the sequel we intend to quantify this fact.

First we prove that if K is large enough then the lengths of the elements of S_0 are large, too. Write $\omega = \cos\alpha + i\sin\alpha$. Let $0 \le j < p/t$. Then

$$|c_{jt}|^2 = |a_{jt} + \bar{\omega} a_{jt+1}|^2 = |a_{jt} + a_{jt+1}\cos\alpha|^2 + |a_{jt+1}\sin\alpha|^2 \ge ||a_{jt}| - |a_{jt+1}\cos\alpha||^2 + |a_{jt+1}\sin\alpha|^2.$$

If $||a_{jt}| - |a_{jt+1}\cos\alpha|| \ge K/4$, then we immediately get $|c_{jt}| \ge K/4$. In the opposite case $|a_{jt+1}\cos\alpha| > |a_{jt} - K/4| \ge K/4$ by (4.1). Hence, as $\sin\alpha \ne 0$ we find

$$|c_{jt}| \ge |a_{jt+1}\sin\alpha| \ge |\tan\alpha| K/4 > k_1 K$$

with a constant k_1, which depends only on k.

Let $0 \le u < t$, then using Lemma 3.1 we get

$$|c_{jt} - (-\omega)^u c_{jt+u}| = |\sum_{\ell=1}^{u} (-\omega)^{\ell-1} b_{jt+\ell-1}| \le u,$$

hence

$$||c_{jt}| - |c_{jt+u}|| \le u \le t.$$

Let finally $0 \le u < v < t$. Then using the former inequalities we obtain

$$\begin{aligned} |c_{jt+u} - c_{jt+v}| &= |c_{jt+u} - (-\omega)^u c_{jt} + (-\omega)^u c_{jt} - (-\omega)^v c_{jt} + (-\omega)^v c_{jt} - c_{jt+v}| \\ &\ge |(-\omega)^u - (-\omega)^v||c_{jt}| - |c_{jt+u} - (-\omega)^u c_{jt}| - |(-\omega)^v c_{jt} - c_{jt+v}| \\ &\ge |(-\omega)^u - (-\omega)^v| K_1 - 2t \\ &\ge k_2 K, \end{aligned}$$

with a constant k_2, which depends only on k.

On the other hand

$$|c_{jt+u} - c_{jt+v}|^2 \le |a_{jt+u} - a_{jt+v} + (a_{jt+u+1} - a_{jt+v+1})\cos\alpha|^2 + |(a_{jt+u+1} - a_{jt+v+1})\sin\alpha|^2.$$

If the second summand is larger than d^2, then we are done. Otherwise

$$|a_{jt+u+1} - a_{jt+v+1}| \le d/|\sin\alpha|,$$

hence

$$|a_{jt+u+1} - a_{jt+v+1}| \ge k_2 K - d|\cot\alpha|,$$

which is larger than d if K is large enough. □

Now we show that the second statement is not empty for a wide class of the parameter λ. More precisely we prove:

Theorem 4.3. *Let k be a positive integer such that $2k+1$ is a prime. Put $\lambda = 2\cos\frac{\pi}{2k+1}$. Then there exist infinitely many $a \in \mathbb{Z}$ such that the λ-orbit with starting values $a_0 = a_1 = a$ has length $2k+1$. Moreover the structure of these orbits is $a_1 a_2 \dots a_k a_{k+1} a_k \dots a_1$.*

To prove this theorem we need some lemma.

Lemma 4.4. *Let $p_0(x) = -1, p_1(x) = 1$ and $p_{n+2}(x) = x p_{n+1}(x) - p_n(x)$ for $n \ge 1$. Then*

$$p_n(x) = \frac{(1+X_2)X_1^n - (1+X_1)X_2^n}{X_1 - X_2},$$

where $X_1 = (x + \sqrt{x^2-4})/2, X_2 = (x - \sqrt{x^2-4})/2$.

This is a well known relation in the theory of second order recurrences (see e.g. [**??**]). The next lemma is crucial for the proof of Theorem 4.3.

Lemma 4.5. *Let $k \ge 1$ and $\lambda = 2\cos\frac{\pi}{2k+1}$. Then*

(i) *$2p_k(\lambda) - \lambda p_{k+1}(\lambda) = 0$ and*
(ii) *$p_{k+1}(\lambda) = 2\sum_{j=0}^{k-1}(-1)^j p_{k-j}(\lambda)$.*

Proof. Let $\omega = \cos\frac{\pi}{2+1} + i\sin\frac{\pi}{2+1}$. Then we have $\lambda = \omega + \bar{\omega}, \omega\bar{\omega} = 1$ and $\omega^{2k+1} = -1$. Specializing $x = \lambda$ in Lemma 4.4 we get $X_1 = \omega, X_2 = \bar{\omega}$. After these preparation we are in the position to prove the statements.
(i) Using the above relations we have:

$$\begin{aligned}
2p_k(\lambda) - \lambda p_{k+1}(\lambda) &= 2\frac{(1+\bar{\omega})\omega^k - (1+\omega)\bar{\omega}^k}{\omega - \bar{\omega}} - (\omega + \bar{\omega})\frac{(1+\bar{\omega})\omega^{k+1} - (1+\omega)\bar{\omega}^{k+1}}{\omega - \bar{\omega}} \\
&= \frac{1}{\omega - \bar{\omega}}\big((1+\bar{\omega})\omega^k(2 - (\omega+\bar{\omega})\omega) - (1+\omega)\bar{\omega}^k(2 - (\omega+\bar{\omega})\bar{\omega})\big) \\
&= \frac{(\omega+1)(\bar{\omega}+1)}{\omega - \bar{\omega}}\big(\bar{\omega}^k(\bar{\omega}-1) - \omega^k(\omega-1)\big) \\
&= \frac{(\omega+1)(\bar{\omega}+1)}{\omega^{k+1}(\omega - \bar{\omega})}\big((1-\omega) - \omega^{2k+1}(\omega-1)\big) = 0.
\end{aligned}$$

Now we turn to prove (ii).

$$\begin{aligned}
2\sum_{j=0}^{k-1}(-1)^j p_{k-j}(\lambda) &= 2\sum_{j=0}^{k-1}(-1)^j\frac{(1+\bar{\omega})\omega^{k-j} - (1+\omega)\bar{\omega}^{k-j}}{\omega - \bar{\omega}} \\
&= 2\frac{(\bar{\omega}+1)\omega^k}{\omega - \bar{\omega}}\sum_{j=0}^{k-1}(-\omega)^{-j} - 2\frac{(\omega+1)\bar{\omega}^k}{\omega - \bar{\omega}}\sum_{j=0}^{k-1}(-\bar{\omega})^{-j} \\
&= 2\frac{\omega^k - \bar{\omega}^k}{\omega - \bar{\omega}} \\
&= p_{k+1}(\lambda).
\end{aligned}$$

□

By the elementary theory of cyclotomic fields λ is an algebraic number of degree $\varphi(2k+1)/2 = k$, where $\varphi(.)$ denotes Euler's totient function (see e.g. [5]).

Corollary 4.6. *If $2k+1$ is a prime and $\lambda = 2\cos\frac{\pi}{2k+1}$ then the real numbers $p_2(\lambda), \ldots, p_{k+1}(\lambda)$ are linearly independent over $\mathbb{Q}$.*

Proof. As $p_j(\lambda)$ is a rational expression of λ of degree $j-1$, the real numbers $p_1(\lambda), \ldots, p_k(\lambda)$ are linearly independent over $\mathbb{Q}$. By Lemma 4.5 (ii) we may replace $p_1(\lambda) = 1$ by $p_{k+1}(\lambda)$ getting again a linearly independent system. □

Lemma 4.7. *Let $a \in \mathbb{Z}$ and*

$$a_j = (-1)^{j-1}\lfloor ap_j(\lambda)\rfloor, j = 1, \ldots, k+1. \tag{4.2}$$

Assume that

$$0 \le (-1)^{j-1}\left(\{ap_{j-1}(\lambda)\} - \lambda\{ap_j(\lambda)\} + \{ap_{j+1}(\lambda)\}\right) < 1, j = 2, \ldots, k, \tag{4.3}$$

$$0 \le (-1)^k\left(2\{ap_k(\lambda)\} - \lambda\{ap_{k+1}(\lambda)\}\right) < 1. \tag{4.4}$$

Then $a_1a_2\ldots a_ka_{k+1}a_k\ldots a_1$ is a λ-orbit.

Proof. It is clear that $a_1 = a$. As

$$a_1(\lambda+1) + a_2 = a(\lambda+1) - \lfloor a(\lambda+1)\rfloor$$

lies in the interval $[0,1)$ the integers a_1, a_1, a_2 satisfy (1.1). Let $2 \le j \le k$. Then

$$\begin{aligned}
a_{j-1} + \lambda a_j + a_{j+1} &= (-1)^{j-2}\lfloor ap_{j-1}(\lambda)\rfloor + \lambda(-1)^{j-1}\lfloor ap_j(\lambda)\rfloor + (-1)^j\lfloor ap_{j+1}(\lambda)\rfloor \\
&= (-1)^j a\left(p_{j-1}(\lambda) - \lambda p_j(\lambda) + p_{j+1}(\lambda)\right) \\
&+ (-1)^{j-1}\left(\{ap_{j-1}(\lambda)\} - \lambda\{ap_j(\lambda)\} + \{ap_{j+1}(\lambda)\}\right).
\end{aligned}$$

The first summand is zero by the definition of the polynomials $p_j(x)$ and the second summand lies in the interval $[0,1)$ by the assumption, hence the integers a_{j-1}, a_j, a_{j+1} satisfy (1.1). Hence, as

$a_1a_2\dots a_ka_{k+1}a_k\dots a_1$ is symmetrical to a_{k+1}, to prove that it is a λ-orbit we have to show that a_k, a_{k+1}, a_k satisfy (1.1) too. We have

$$\begin{aligned} 2a_k + \lambda a_{k+1} &= 2(-1)^{k-1}\lfloor ap_k(\lambda)\rfloor + \lambda(-1)^k\lfloor ap_{k+1}(\lambda)\rfloor \\ &= (-1)^{k-1}a\,(2p_k(\lambda) - \lambda p_{k+1}(\lambda)) + (-1)^k\,(2\{ap_k(\lambda)\} - \lambda\{ap_{k+1}(\lambda)\}) \,. \end{aligned}$$

Here the first summand is zero by Lemma 4.5 (i) and the second is lying in the interval $[0,1)$ by the assumption. The lemma is proved. □

Lemma 4.8. *Let $D_n(x)$ denote the determinant of the $n\times n$ matrix*

$$M_n(x) = \begin{pmatrix} x & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & x & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & x & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & x & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & x \end{pmatrix}.$$

Then $D_1(x) = x, D_2(x) = x^2 - 2$ and $D_{n+2}(x) = xD_{n+1}(x) - D_n(x)$ hold for any $n \geq 1$. Moreover

$$D_n(x) = X_1^n + X_2^n,$$

where X_1, X_2 are defined in Lemma 4.4.

Proof. The expressions for $D_1(x)$ and $D_2(x)$ are obvious. The recurrence relation follows, if we compute $D_{n+2}(x)$ by the first row. The "analytical" expression is an immediate consequence of the recurrence relation. □

Corollary 4.9. *If ω is a primitive $4k+2$-th root of unity and $\lambda = 2\Re\omega$ then $D_k(\lambda) \neq 0$, i.e. $M_k(\lambda)$ is invertible.*

Proof. We have

$$D_k(\lambda) = \omega^k + \bar\omega^k = \frac{\omega^{2k}+1}{\omega^k} = \frac{-\bar\omega+1}{\omega^k} \neq 0$$

by Lemma 4.8 and as $\omega^{2k+1} = -1$. □

Proof of Theorem 4.3. Consider the system of inequalities (I)

$$\begin{aligned} 0 &\leq (-1)^{j-1}(x_{j-1} - \lambda x_j + x_{j+1}) < 1, j = 2,\dots,k \\ 0 &\leq (-1)^k(2x_k - \lambda x_{k+1}) < 1, \end{aligned}$$

where $x_1 = 0$. Put

$$S = \{(x_2,\dots,x_{k+1}) \in \mathbb{R}^k, (x_2,\dots,x_{k+1}) \text{ satisfies } (I)\} \cap [0,1)^k.$$

A ball B with small enough positive radius around the point $p = (p_2,\dots,p_{k+1})$ with $p_{2j} = 1/2, j = 1,\dots,\lfloor(k+1)/2\rfloor$ and $p_{2j+1} = 1/4, j = 1,\dots,\lfloor(k+1)/2\rfloor$ is contained in the set S, because $1 < \lambda < 2$.
The real numbers $p_2(\lambda),\dots,p_{k+1}(\lambda)$ are by Corollary 4.6 linearly independent over $\mathbb{Q}$. Hence the set

$$\{(\{ap_2(\lambda)\},\dots,\{ap_{k+1}(\lambda)\})|a\in\mathbb{Z}\}$$

is everywhere dense in $[0,1)^k$ (see e.g. [3], Theorem 1, §3.). Thus its intersection with S is an infinite set, moreover these points satisfy the inequalities (4.3) and (4.4). Notice that $\{ap_1(\lambda)\} = 0$. By Lemma 4.7 there exist infinitely many $a \in \mathbb{Z}$ such that the λ-orbits with starting values a, a are periodic of length $2k+1$. □

Remark 4.10. (i) With some effort one could prove similar results for $4k+2$-th primitive roots of unity which do not lie in the first quadrant, and for $2k+1$-th roots of unity. The minimal period length is usually $2k+1$ or $4k+2$. The following question seems to be more challenging: do there exist infinitely many $2\Re\omega$-orbits with arbitrary large minimal period length p?

(ii) Assume that $\omega \in \mathbb{C}$ is not a root of unity and let $(a_n)_{n\in\mathbb{Z}}$ be a periodic orbit belonging to $\lambda = 2\Re\omega$ with minimal period length p. Setting for $a = (a_n)_{n\in\mathbb{Z}}$

$$\delta_j(a) = \max\{|a_n - a_{n+j}| \,|\, n \in \mathbb{Z}\}$$

$$\text{and} \quad d(a) = \min\{\delta_j(a) \,|\, 0 < j < p, (p,j) = 1\}$$

it is easy to see by using the same argument as in the beginning of the proof of the last theorem that the set $\{(a_n, a_{n+1}) \,|\, n \in \mathbb{Z}\}$ is d-connected. In contrast to Theorem 4.2 this d depends not only on ω, but also on the actual sequence.

5. λ is the golden ratio,i.e. $\lambda = \frac{1+\sqrt{5}}{2}$

The purpose of this section is prove the conjecture in some case, when $\pm\lambda$ is two times the real part of a root of unity. In the following table we display twice of the real parts of roots of unity of low order.

order	1	2	3	4	5	5
λ	2	-2	-1	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$

Theorem 5.1. *Every orbit belonging to* $\lambda = \frac{1+\sqrt{5}}{2}$ *is periodic.*

Proof. We have

$$a_{n+1} = -a_{n-1} - \lambda a_n + \{\lambda a_n\}$$

$$a_n + \lambda a_{n+1} = -\lambda a_{n-1} - \lambda a_n + \lambda\{\lambda a_n\} = -\lfloor\lambda a_{n-1}\rfloor - \{\lambda a_{n-1}\} - \lfloor\lambda a_n\rfloor + \frac{1}{\lambda}\{\lambda a_n\}$$

hence

$$a_{n+2} = \lambda a_{n-1} - \{\lambda a_{n-1}\} + \lambda a_n - \{\lambda a_n\} + c \text{ with } c = \begin{cases} 1 & \text{if } \{\lambda a_{n-1}\} > \frac{1}{\lambda}\{\lambda a_n\} \\ 0 & \text{if } \{\lambda a_{n-1}\} \le \frac{1}{\lambda}\{\lambda a_n\} \end{cases}$$

$$a_{n+1} + \lambda a_{n+2} = \lambda a_{n-1} - \lambda\{\lambda a_{n-1}\} + a_n - \frac{1}{\lambda}\{\lambda a_n\} + \lambda c$$

$$= \lfloor\lambda a_{n-1}\rfloor - \frac{1}{\lambda}\{\lambda a_{n-1}\} + a_n - \frac{1}{\lambda}\{\lambda a_n\} + \lambda c$$

$$a_{n+3} = -\lambda a_{n-1} + \{\lambda a_{n-1}\} - a_n + c' \text{ with } c' = \begin{cases} 1 & \text{if } \{\lambda a_{n-1}\} \le \frac{1}{\lambda}\{\lambda a_n\} \\ 0 & \text{if } \{\lambda a_{n-1}\} > \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} > 1 \\ -1 & \text{if } \{\lambda a_{n-1}\} > \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} \le 1 \end{cases}$$

$$a_{n+2} + \lambda a_{n+3} = -a_{n-1} + \frac{1}{\lambda}\{\lambda a_{n-1}\} - \{\lambda a_n\} + c + \lambda c'$$

$$a_{n+4} = a_{n-1} + d_{n-1} \text{ with } d_{n-1} = \begin{cases} 1 & \text{if } \{\lambda a_{n-1}\} > \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} \le 1 \\ 0 & \text{if } \frac{1}{\lambda}\{\lambda a_n\} < \{\lambda a_{n-1}\} < \lambda\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} > 1 \\ & \text{or } \{\lambda a_{n-1}\} \le \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_n\} > \frac{1}{\lambda} + \frac{1}{\lambda}\{\lambda a_{n-1}\} \\ -1 & \text{if } \{\lambda a_{n-1}\} \le \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_n\} \le \frac{1}{\lambda} + \frac{1}{\lambda}\{\lambda a_{n-1}\} \\ & \text{or } \{\lambda a_{n-1}\} \le \lambda\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} > 1 \end{cases}$$

The condition $\{\lambda a_{n-1}\} \le \frac{1}{\lambda}\{\lambda a_n\}$ in the third line of the expression for d_{n-1} is not necessary since it is implied by $\{\lambda a_n\} > \frac{1}{\lambda} + \frac{1}{\lambda}\{\lambda a_{n-1}\}$. Finally we have

$$a_{n+3} + \lambda a_{n+4} = \{\lambda a_{n-1}\} - a_n + c' + \lambda d_{n-1}$$

$$a_{n+5} = a_n + d_n \text{ with } d_n = \begin{cases} 1 & \text{if } \{\lambda a_{n-1}\} \ge \lambda\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} > 1 \\ & \text{or } \{\lambda a_{n-1}\} \le \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_n\} \le \frac{1}{\lambda} + \frac{1}{\lambda}\{\lambda a_{n-1}\} \\ 0 & \text{if } \{\lambda a_{n-1}\} > \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} \le 1, \{\lambda a_{n-1}\} < \frac{1}{\lambda^2} \\ & \text{or } \frac{1}{\lambda}\{\lambda a_n\} < \{\lambda a_{n-1}\} < \lambda\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} > 1 \\ -1 & \text{if } \{\lambda a_{n-1}\} > \frac{1}{\lambda}\{\lambda a_n\}, \{\lambda a_{n-1}\} + \{\lambda a_n\} \le 1, \{\lambda a_{n-1}\} \ge \frac{1}{\lambda^2} \\ & \text{or } \{\lambda a_n\} > \frac{1}{\lambda} + \frac{1}{\lambda}\{\lambda a_{n-1}\} \end{cases}$$

(In the first line, $1 < \{\lambda a_{n-1}\} + \{\lambda a_n\} \le \lambda\{\lambda a_{n-1}\}$ implies $\{\lambda a_{n-1}\} > \frac{1}{\lambda}$.)

Therefore $|d_n| = |a_{n+5} - a_n| \le 1$ for all $n \in \mathbb{Z}$.

The Fibonacci numbers are given by

$$F_j = \frac{1}{\sqrt{5}}\lambda^j - \frac{1}{\sqrt{5}}\left(-\frac{1}{\lambda}\right)^j,$$

$$\lambda F_j = \frac{1}{\sqrt{5}}\lambda^{j+1} + \frac{\lambda^2}{\sqrt{5}}\left(-\frac{1}{\lambda}\right)^{j+1} = F_{j+1} + \frac{\lambda^2+1}{\sqrt{5}}\left(-\frac{1}{\lambda}\right)^{j+1} = F_{j+1} + \frac{(-1)^{j+1}}{\lambda^j}$$

Suppose $a_n = F_{2m-1}$ (with $n \geq 2, m \geq 2$) and $a_j \leq F_{2m-1}$ for all $j \in \{0, \dots, n-1\}$. Then we have $\lambda a_{n-1} \geq -a_{n-2} - a_n \geq -2F_{2m-1}$, thus $-F_{2m} < a_{n-1} \leq F_{2m-1}$. Since the F_j are the denominators of the best approximations of λ (see e.g. [**?**]), we have

$$\{\lambda a_n\} \leq \{\lambda a_{n-1}\} \leq 1 - \{\lambda a_n\}, \text{ hence } d_n = a_{n+5} - a_n \in \{-1, 0\}.$$

If we choose m such that $a_j \leq F_{2m-1}$ for all $j \in \{0, \dots, 6\}$, then $a_5, \dots, a_9 \leq F_{2m-1}$ and, inductively, $a_{5k}, \dots, a_{5k+4} \leq F_{2m-1}$ for all $k \geq 0$. An application of Proposition 2.5 concludes the proof. □

6. The cases $\lambda = \pm\sqrt{2}$

For an orbit $(a_n)_{n\in\mathbb{Z}}$ belonging to λ we have the relation

$$b_{n-1} - \frac{2}{\lambda}b_n + b_{n+1} = a_{n-1} + (\lambda - \frac{2}{\lambda})a_n + (\lambda - \frac{2}{\lambda})a_{n+2} + a_{n+3} \ \ (n \in \mathbb{Z}).$$

If $\lambda = \pm\sqrt{2}$ then in the above equation the middle terms vanish and we get

$$b_{n-1} - \frac{2}{\lambda}b_n + b_{n+1} = a_{n-1} + a_{n+3}. \tag{6.1}$$

Hence for $\lambda = \pm\sqrt{2}$ it is convenient to introduce another sequence of integers $(d_n)_{n\in\mathbb{Z}}$ by

$$d_n = a_n + a_{n+4}.$$

Lemma 6.1. *If $\lambda = \sqrt{2}$ then $d_n \in \{-1, 0, 1\}$, moreover $(d_{n-1}, d_n, d_{n+1}) \in \{(1,-1,1), (0,0,0), (1,0,0), (0,0,1), (0,1,0), (-1,1,0), (0,1,-1)\}$. If $\lambda = -\sqrt{2}$ and $\{a_n\} \neq \{0\}$ then $d_n \in \{1, 2, 3\}$, moreover $(d_{n-1}, d_n, d_{n+1}) \in \{(1,1,1), (1,1,2), (2,1,1), (1,2,2), (1,2,3), (3,2,1), (2,2,1), (2,2,2), (2,3,3), (3,3,2), (3,3,3)\}$.*

Proof. By (6.1) we have

$$d_{n-1} + \lambda d_n + d_{n+1} = b_{n-1} + b_{n+3}. \tag{6.2}$$

If $\lambda = \sqrt{2}$ then by (1.1) and (6.1) we have $-\sqrt{2} \leq d_n < 2$ and as d_n is an integer we get the assertion for d_n. As the right hand side of 6.2 is at least zero and less then 2, we get that if $d_n = -1$ then d_{n-1} and d_{n+1} have to be 1. If $d_n = 0$ then d_{n-1} and d_{n+1} can be zero or one, but both cannot be one. If $d_n = 1$ then d_{n-1} and d_{n+1} can be zero or minus one, but both cannot be minus one. To exclude the pattern $(1, 1, -1)$ needs a more careful analysis. In that case $b_{n-1} + b_{n+3} = \sqrt{2}$ and $b_{n+1} - \sqrt{2}b_{n+2} + b_{n+3} = -1$ hold. Thus $b_{n+1} = -\sqrt{2} - 1 + \sqrt{2}b_{n+2} + b_{n-1}$, which contradicts $b_{n+1} \geq 0$.

Similarly, if $\lambda = -\sqrt{2}$ then by (1.1) and (6.1) we have $0 \leq d_n < 2 + \sqrt{2}$. If $d_n = 0$ then $b_{n-1} = b_n = b_{n+1} = 0$ which implies $a_k = 0$ for $k \geq n$. Hence, if the orbit is not the 0−sequence and as d_n is an integer we get the assertion for d_n. As the right hand side of (6.2) is at least zero and less then 2, we get that if $d_n = 1$ then neither d_{n-1} nor d_{n+1} can be 3. If $d_n = 2$ then both d_{n-1} and d_{n+1} cannot be 1 or 3. Finally, if $d_n = 3$ then neither d_{n-1} nor d_{n+1} can be 1 and $(2, 3, 2)$ is excluded, too. □

6.1. **The special case $\lambda = \sqrt{2}$ and $(d_n, d_{n+1}, d_{n+2}) = (0,0,0)$.** The case in the title seems to be the simplest, but as we shall see later it is not simple at all. We assume throughout this subsection $\lambda = \sqrt{2}$ and $n = 1$, i.e. $(d_1, d_2, d_3) = (0,0,0)$. Assume further that our orbit looks like

$$\dots, a_1, a_2, a_3, x, -a_1, -a_2, -a_3, y, \dots$$

The inequalities

$$0 \le a_1 + \sqrt{2}a_2 + a_3 < 1$$
$$0 \le -a_1 - \sqrt{2}a_2 - a_3 < 1$$

imply $a_1 + \sqrt{2}a_2 + a_3 = 0$, from which we get $a_2 = 0$ and $a_3 = -a_1$. Hence the sequence has the much more special shape

$$\dots, a_1, 0, -a_1, x, -a_1, 0, a_1, y, z, \dots$$

To obtain a relation between x and y consider the inequalities

$$0 \le -a_1\sqrt{2} + x < 1 \tag{6.3}$$
$$0 \le -2a_1 + x\sqrt{2} < 1 \tag{6.4}$$
$$0 \le a_1\sqrt{2} + y < 1 \tag{6.5}$$
$$0 \le a_1 + y\sqrt{2} + z < 1. \tag{6.6}$$

The first implies that if $x = 0$, then $a_1 = 0$, i.e. $(a_n)_{n\in\mathbb{Z}}$ is the $0-$sequence. In the sequel we assume $x \ne 0$. Dividing (6.4) by $\sqrt{2}$ we obtain

$$0 < -a_1\sqrt{2} + x < \frac{\sqrt{2}}{2}. \tag{6.7}$$

The last inequality implies

$$1 - \frac{\sqrt{2}}{2} < a_1\sqrt{2} - x + 1 < 1,$$

hence $y = -x + 1$.

Another consequence of (6.4) is that

$$0 < a_1 + (-x+1)\sqrt{2} + a_1 + 1 - \sqrt{2} < 1$$

keeping in mind $y \ne 1$. This means that we have two possibilities for z:

(a) $z = a_1$ or
(b) $z = a_1 - 1$.

Case (a) occurs if and only if

$$0 \le 2a_1 + (-x+1)\sqrt{2} < 1,$$

i.e. if

$$-1 \le a_1\sqrt{2} - x < \frac{\sqrt{2}}{2} - 1.$$

Combining this with (6.7) we obtain that (a) holds if and only if

$$1 - \frac{\sqrt{2}}{2} < a_1\sqrt{2} - (x-1) < \frac{\sqrt{2}}{2},$$

i.e. if the fractional part of $a_1\sqrt{2}$ belongs to the interval $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Moreover our orbit has the shape

$$a_1, 0, -a_1, x, -a_1, 0, a_1, -x+1, a_1, \dots \tag{6.8}$$

As the starting pattern $a_1, -x+1, a_1$ can only be continued symmetrically the next member of the sequence is 0, thus the orbit is periodic with minimal period length 8. We summarize the result as follows.

Theorem 6.2. *Let $a_1 \in \mathbb{Z} \setminus \{0\}$ be such that the fractional part of $a_1\sqrt{2}$ belongs to the interval $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Then the $\sqrt{2}$-orbit starting with $a_1, 0$ admits the period (6.8) with $x = -\lfloor -a_1\sqrt{2} \rfloor$ of length 8. The corresponding sequence $(d_n)_{n\in\mathbb{Z}}$ is $\overline{0001}$. There exist infinitely many a_1 satisfying the above assumptions.*

Proof. It remains to prove the last statement. This follows from the fact that the sequence $(\{n\sqrt{2}\})_{n\in\mathbb{Z}}$ is uniformly distributed in the interval $[0,1)$. □

Remark 6.3. Case (b) is much more complicated. For example $-2,0,2,-2,1,1,-2,2,0,-2,3$ is periodic of length 11. We will show later that this is the only orbit of period length 11.

The next statement is in some sense the converse of Theorem 6.2.

Theorem 6.4. *Up to the* $0-$*sequence every orbit belonging to* $\sqrt{2}$ *and having period length* 8 *has to be of the form described in Theorem 6.2.*

Proof. Assume that the period length of $(a_n)_{n\in\mathbb{Z}}$ is 8, hence the period length of $(d_n)_{n\in\mathbb{Z}}$ is four. We distinguish five cases according to the number ν of zeros in the period of $(d_n)_{n\in\mathbb{Z}}$; here we constantly exploit Lemma 6.1.

If $\nu=4$ then $a_n=a$ for all $n\in\mathbb{Z}$ and $0\le a(2+\sqrt{2})<1$ implies $a=0$.

If $\nu=3$ then the fourth member of the period has to be 1, and we are in the situation of Theorem 6.2.

If $\nu=2$ then a neighbor of 0 can only be 0 or 1. Thus, the period is $\overline{0011}$ or $\overline{0101}$, but 011 and 101 are excluded.

If $\nu=1$ then the only four term sequence which can occur is $0,1,-1,1$. Extending this periodically we obtain $0,1,-1,1,0,1,-1,1$ which is not allowed because of the middle subsequence $1,0,1$.

If $\nu=0$ then the period must contain the sequence $1,-1,1$, which can only be extended by a 1. Again continuing periodically the result contains $1,1,1$, which is excluded. □

Corollary 6.5. *Except the* $0-$*sequence there exist no* $\sqrt{2}$*-orbits having minimal period length* $1,2$ *or* 4.

6.2. **Periods of orbits belonging to $\lambda=\pm\sqrt{2}$.** In this subsection we will show that only multiples of 8 can occur as minimal period lengths of orbits belonging to $\lambda=\pm\sqrt{2}$.

Lemma 6.6. *Let* $\lambda=\pm\sqrt{2}$ *and* $n,k\in\mathbb{Z},k>0$. *Then*

$$a_{n+4k}=(-1)^k a_n+\sum_{j=0}^{k-1}(-1)^j d_{n+4(k-j-1)}.$$

Proof. If $k=1$ then $a_{n+4}=-a_n+d_n$, which is true by the definition of d_n. Let now $k\ge 1$. Then

$$\begin{aligned} a_{n+4(k+1)} &= -a_{n+4k}+d_{n+4k}\\ &= -\left((-1)^k a_n+\sum_{j=0}^{k-1}(-1)^j d_{n+4(k-j-1)}\right)+d_{n+4k}\\ &= (-1)^{k+1}a_n+\sum_{j=0}^{k}(-1)^j d_{n+4(k+1-j-1)}, \end{aligned}$$

where we used the definition of d_n and the induction hypothesis. □

Theorem 6.7. *Let* $\lambda=\pm\sqrt{2}$ *and assume that the minimal period length of* $(a_n)_{n\in\mathbb{Z}}$ *is* p, *where* $p=4k+1$ *or* $p=4k-1, k>0$. *Then*

$$|a_n|\le\begin{cases} \frac{k+3}{2(2-\sqrt{2})} & \text{, if } k \text{ is odd and } \lambda=\sqrt{2},\\ \frac{2k+1}{2+\sqrt{2}} & \text{, if } k \text{ is even and } \lambda=\sqrt{2},\\ \frac{3k+4}{2+\sqrt{2}} & \text{ if } k \text{ is odd and } \lambda=-\sqrt{2},\\ \frac{2k+1}{2-\sqrt{2}} & \text{, if } k \text{ is even and } \lambda=-\sqrt{2}. \end{cases}$$

Proof. We start with the case $p = 4k+1$. By periodicity and Lemma 6.6 we get

$$a_1 = a_{p+1} = a_{4k+2} = (-1)^k a_2 + M_1, \;\; a_2 = a_{p+2} = a_{4k+3} = (-1)^k a_3 + M_2, \tag{6.9}$$

where

$$M_1 = \sum_{j=0}^{k-1} (-1)^j d_{2+4(k-j-1)}, \;\; M_2 = \sum_{j=0}^{k-1} (-1)^j d_{3+4(k-j-1)}.$$

Equation (6.9) implies

$$a_3 = (-1)^k a_2 + (-1)^{k+1} M_2.$$

Inserting this and the formula for a_1 given in (6.9) into (1.1) we get

$$0 \le (-1)^k a_2 + M_1 + \lambda a_2 + (-1)^k a_2 + (-1)^{k+1} M_2 < 1,$$

which implies

$$-M_1 + (-1)^k M_2 \le (\lambda + 2(-1)^k) a_2 < 1 - M_1 + (-1)^k M_2. \tag{6.10}$$

Now we have to estimate $-M_1 + (-1)^k M_2$. We have

$$-M_1 + (-1)^k M_2 = \sum_{j=0}^{k-1} (-1)^{j+1} (d_{2+4(k-j-1)} + (-1)^{k+1} d_{3+4(k-j-1)}) \tag{6.11}$$

and distinguish four cases.

Case I: $\lambda = \sqrt{2}, k$ odd. By Lemma 6.1 the only possibilities for $(d_{2+4(k-j-1)}, d_{3+4(k-j-1)})$ are $(0,1), (1,0), (0,0), (1,-1)$ and $(-1,1)$, hence

$$0 \le d_{2+4(k-j-1)} + d_{3+4(k-j-1)} \le 1,$$

which implies

$$|M_1 + M_2| \le \frac{k+1}{2},$$

and finally by (6.10)

$$|a_2| \le \frac{k+3}{2(2-\sqrt{2})}.$$

As we can choose the starting point of the orbit arbitrarily, the same inequality holds for all members of the orbit.

Case II: $\lambda = \sqrt{2}, k$ even. Then

$$-2 \le d_{2+4(k-j-1)} - d_{3+4(k-j-1)} \le 2,$$

which implies

$$|M_1 - M_2| \le 2k,$$

and finally

$$|a_2| \le \frac{2k+1}{2+\sqrt{2}}.$$

Case III: $\lambda = -\sqrt{2}, k$ odd. Here

$$0 \le d_{2+4(k-j-1)} + d_{3+4(k-j-1)} \le 6,$$

which implies

$$|M_1 + M_2| \le 6\frac{k-1}{2} + 6 = 3(k+1),$$

and finally

$$|a_2| \le \frac{3k+4}{2+\sqrt{2}}.$$

Case IV: $\lambda = -\sqrt{2}, k$ even. By Lemma 6.1 the pair $(d_{2+4(k-j-1)}, d_{3+4(k-j-1)})$ cannot be $(3,1)$ and $(1,3)$, hence

$$-2 \le d_{2+4(k-j-1)} - d_{3+4(k-j-1)} \le 2,$$

which implies

$$|M_1 - M_2| \le 2k,$$

and finally

$$|a_2| \le \frac{2k+1}{2-\sqrt{2}}.$$

Now we turn to the case $p = 4k-1$. By periodicity we obtain

$$a_2 = a_{p+2} = a_{4k+1} = (-1)^k a_1 + M_1, \;\; a_3 = a_{p+3} = a_{4k+2} = (-1)^k a_2 + M_2, \tag{6.12}$$

where

$$M_1 = \sum_{j=0}^{k-1} (-1)^j d_{1+4(k-j-1)}, \;\; M_2 = \sum_{j=0}^{k-1} (-1)^j d_{2+4(k-j-1)}.$$

Equation (6.12) implies

$$a_1 = (-1)^k a_2 + (-1)^{k+1} M_1.$$

On this way we obtain again (6.10), where M_1 and M_2 have different meaning, but the estimates remain true, because we used only properties of (consecutive) members of the sequence d_n. □

Theorem 6.8. *Let $\lambda = \pm\sqrt{2}$ and assume that the minimal period length of $(a_n)_{n\in\mathbb{Z}}$ is $4k+2$, $k > 0$. Then*

$$|a_n| \le \begin{cases} \frac{k+3}{\sqrt{2}} & \text{, if } k \text{ is odd,} \\ k+1 & \text{, if } k \text{ is even.} \end{cases}$$

Proof. It is similar to the proof of Theorem 6.7. □

The previous theorems imply immediately the following corollary.

Corollary 6.9. *Let $\lambda = \pm\sqrt{2}$ and p be a positive integer, which is not divisible by 8. Then there exist only finitely many λ-orbits $(a_n)_{n\in\mathbb{Z}}$ with minimal period length p.*

Corollary 6.10. *If $\lambda = \sqrt{2}$ then apart from the zero and if $\lambda = -\sqrt{2}$ then apart from the zero and one sequences there exist no λ-orbits of minimal period length less then 8.*

Proof. Let p denote the period length of $(a_n)_{n\in\mathbb{Z}}$. Put $p = 6$. Then $|a_n| \le \frac{4}{\sqrt{2}}$ by Theorem 6.8 and as a_n are integers we get $|a_n| \le 2$. Assume that $(a_n)_{n\in\mathbb{Z}}$ is not the 0−sequence, then it contains at least one positive member: This is obviously true if $\lambda = \sqrt{2}$. If $\lambda = -\sqrt{2}$ and $a_1 < 0$, then $a_5 = -a_1 + d_1$, where $d_1 \ge 1$. Thus $a_5 > 0$. It remains to test the sequences with starting members $a_1, a_2; a_1 = 1, 2, |a_2| \le 2$, but apart from the exceptional cases given in the theorem, the period length all of the sequences are at least eight. Note that the same argument works for $p = 12$, too.

Let $\lambda = \sqrt{2}$. We proved in Corollary 6.5 that $p \ne 1, 2, 4$ and the above argument implies $p \ne 3, 6$. If $p = 5$ then $|a_n| \le \frac{4}{2(2-\sqrt{2})}$, hence $|a_n| \le 3$, and if $p = 7$ then $|a_n| \le \frac{3}{(2+\sqrt{2})}$, i.e $|a_n| = 0$. We can easily test in the first case the candidates, but found only periodic sequences of minimal period length at least 8.

In the sequel let $\lambda = -\sqrt{2}$. If $p = 4$ then $|a_n| \le 1$ by Theorem 6.11 and we are done. Similarly, if $p = 5$ then $|a_n| \le 2$, but these cases were already considered in the case $p = 6$. Finally, if $p = 7$ then $|a_n| \le 8$ by Theorem 6.7. Considering however $p = 14$ Theorem 6.8 implies $|a_n| \le 4$. Testing the candidates as before we found only periodic sequences of minimal period length at least 8. □

Theorem 6.11. *Let $\lambda = \pm\sqrt{2}$ and assume that the minimal period length of the λ-orbit $(a_n)_{n\in\mathbb{Z}}$ is $4k$ with k is odd. Then*

$$|a_n| \le \frac{k+2}{2}.$$

Proof. Using again the periodicity and Lemma 6.6 we get

$$a_1 = a_{4k+1} = -a_1 + M_1,$$

where

$$M_1 = \sum_{j=0}^{k-1} (-1)^j d_{1+4(k-j-1)}.$$

The first equation implies

$$|a_1| = |M_1|/2$$

and as $|M_1| \le k$, if $\lambda = \sqrt{2}$ and $|M_1| \le k+2$, if $\lambda = -\sqrt{2}$ we obtain the statement. □

We conclude the section by exhibiting two infinite series of short period lengths.

Proposition 6.12. *Let $\lambda = -\sqrt{2}$ and assume that the integers a_1, a_2 satisfy the inequality*

$$\sqrt{2} - 1 \le a_1 - \sqrt{2}a_2 \le 2 - \sqrt{2}. \tag{6.13}$$

Then the λ-orbit generated by a_1, a_2 is

$$a_1, a_2, 0, -a_2, -a_1+1, -a_2+1, 1, a_2+1, \tag{6.14}$$

i.e it has period length 8. There are infinitely many λ-orbits of period length 8.

Proof. The elements (6.14) define a λ-orbit if and only if the following inequalities hold.

$$0 \le a_1 - \sqrt{2}a_2 < 1,\ \ 0 \le -a_1 + \sqrt{2}a_2 + 1 < 1,\ \ 0 \le -2a_2 + \sqrt{2}a_1 + 1 - \sqrt{2} < 1,$$

$$0 \le a_1 - \sqrt{2}a_2 + 1 - \sqrt{2} < 1,\ \ 0 \le -a_1 + \sqrt{2}a_2 + 2 - \sqrt{2} < 1,\ \ 0 \le 2a_2 - \sqrt{2}a_1 + 1 < 1.$$

A straightforward computation shows that if (6.13) holds then all of these inequalities hold, too. As $\sqrt{2}$ is irrational there exist infinitely many $a_1, a_2 \in \mathbb{Z}$ which satisfy (6.13). □

Remark 6.13. It is easy to check that if a $-\sqrt{2}$-orbit has period length 8 then the corresponding sequence $(d_n)_{n\in\mathbb{Z}}$ is $\overline{1}, \overline{2}, \overline{3}$ or $\overline{1122}$.

The next example is based on a different value for λ, but the technique of proof is identical to that in Proposition 6.12.

Proposition 6.14. *Let $\lambda = -\sqrt{3}$ and assume that the integers a_1, a_2 satisfy the inequality*

$$0 < a_1 - \sqrt{3}a_2 \le 7 - 4\sqrt{3}.$$

Then the λ-orbit generated by a_1, a_2 is

$$a_1, a_2, 0, -a_2, -a_1+1, -2a_2+2, -a_1+3, -a_2+4, 4, a_2+3, a_1+2, 2a_2+1, \tag{6.15}$$

i.e it has period length 12. There are infinitely many λ-orbits of period length 12.

Proof. The elements (6.15) define a λ-orbit if and only if the following inequalities hold.

$$0 \le a_1 - \sqrt{3}a_2 < 1,\ \ 0 \le \sqrt{3}a_2 - a_1 + 1 < 1,\ \ 0 \le -3a_2 + 2 - \sqrt{3}(-a_1+1) < 1,$$

$$0 \le -2a_1+4-\sqrt{3}(-2a_2+2) < 1,\ \ 0 \le -3a_2+6-\sqrt{3}(-a_1+3) < 1,\ \ 0 \le -a_1+7-\sqrt{3}(-a_2+4) < 1,$$

$$0 \le a_1 + 6 - \sqrt{3}(a_2+3) < 1,\ \ 0 \le 3a_2 + 4 - \sqrt{3}(a_1+2) < 1,\ \ 0 \le 2a_1 + 2 - \sqrt{3}(2a_2+1) < 1,$$

$$0 \le 3a_2 + 1 - \sqrt{3}a_1 < 1.$$

The proof can now be finished analogously as the proof of Proposition 6.12. □

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