

Rational base numeration system for p -adic numbers

Karel Klouda^{1,2}
karel@kloudak.eu

Joint work with C. Frougny

¹FNSPE, CTU in Prague

²LIAFA, Université Paris 7

Digital expansions, dynamics and tilings
April 4-11, 2010, Aussois

Outline

$$\sum_{k=-\ell}^{n, \infty} \frac{a_j}{q} \left(\frac{p}{q}\right)^i,$$

$a_j \in \mathcal{A}_p = \{0, 1, \dots, p-1\}$, $p > q \geq 1$ co-prime

1. Brief introduction to p -adic numbers
2. Representation of positive integers [Akiyama, Frougny, Sakarovitch, 2008]
3. Representation of negative integers
4. Representation of rational numbers
5. Finite representations
6. Representations of p -adic numbers

p -adic absolute value

Definition

Let p be a prime number.

The **p -adic valuation** $v_p : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{R}$ is given by

$$n = p^{v_p(n)} n' \quad \text{with} \quad p \nmid n'.$$

The extension to the set of rational numbers, for $x = \frac{a}{b} \in \mathbb{Q}$

$$v_p(x) = v_p(a) - v_p(b).$$

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The **p -adic absolute value** on \mathbb{Q} :

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-v_p(x)} & \text{otherwise.} \end{cases}$$

p -adic absolute value – examples

Example

$$v_3(6) = 1 \quad \Rightarrow \quad |6|_3 = 3^{-1}, |1/6|_3 = 3^1$$

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$$p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}, q \text{ co-prime to } p \quad \Rightarrow$$

$$\left| \left(\frac{p}{q} \right)^n \right|_r = \begin{cases} r_i^{-j_i n} & \text{if } r = r_i, i \in \{1, 2, \dots, k\}, \\ \geq 1 & \text{otherwise.} \end{cases}$$

\mathbb{Q}_p – the set of p -adic numbers I

The set of p -adic numbers \mathbb{Q}_p is defined as a **completion of \mathbb{Q}** with respect to $|\cdot|_p$.

Lemma

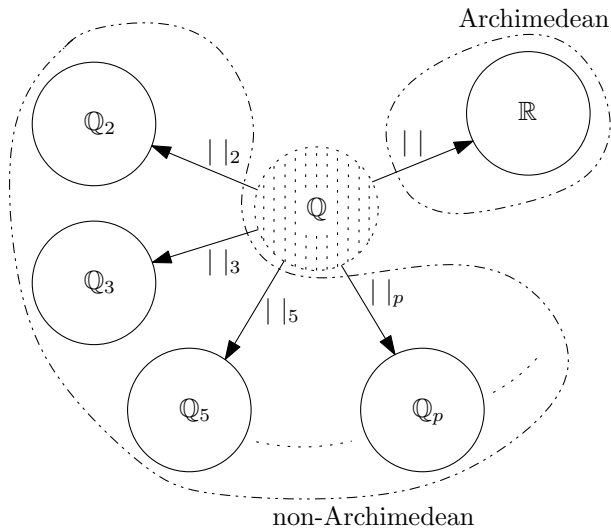
*The p -adic absolute value $|\cdot|_p$ is **non-Archimedean**, i.e., for all $x, y \in \mathbb{Q}_r$ the following holds:*

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

Theorem (Ostrowski)

Every non-trivial absolute value on \mathbb{Q} is equivalent to the classical absolute value $|\cdot|$ or to one of the absolute values $|\cdot|_p$, where p is prime.

\mathbb{Q}_p – the set of p -adic numbers II



Standard representation of p -adic numbers I

Theorem

Every $x \in \mathbb{Q}_p$ can be uniquely written as

$$\begin{aligned}x &= b_{-k_0} p^{-k_0} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k + \cdots \\ &= \sum_{k \geq -k_0} a_k p^k\end{aligned}$$

with $a_k \in \mathcal{A}_p = \{0, 1, \dots, p-1\}$ and $k_0 = -v_p(x)$.

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Algorithm

Let $s \in \mathbb{Z}$, $s \neq 0$. Put $s_0 = s$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by

$$s_i = ps_{i+1} + a_i.$$

Return $\mathbf{a} = \cdots a_2 a_1 a_0$.

Standard representation of p -adic numbers I

Theorem

Let $x \in \mathbb{Q}_p$. Then the standard representation of x is

1. uniquely given,
2. finite if, and only if, $x \in \mathbb{N}$,
3. eventually periodic if, and only if, $x \in \mathbb{Q}$.

MD algorithm

Algorithm (MD algorithm)

Let s be a positive integer. Put $s_0 = s$ and for all $i \in \mathbb{N}$:

$$qs_i = ps_{i+1} + a_i.$$

Return $\frac{p}{q}$ -*expansion* of s : $\langle s \rangle_{\frac{p}{q}} = \cdots a_2 a_1 a_0$.

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$$\begin{aligned} s = s_0 &= s_1 \frac{p}{q} + \frac{a_0}{q} = s_2 \left(\frac{p}{q}\right)^2 + \frac{a_1 p}{q q} + \frac{a_0}{q} = \cdots \\ &\cdots = s_k \left(\frac{p}{q}\right)^k + \sum_{i=1}^{k-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^i = \cdots = \sum_{i=1}^n \frac{a_i}{q} \left(\frac{p}{q}\right)^i \end{aligned}$$

$\frac{p}{q}$ -expansion – properties

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$$L_{\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid w \text{ is } \frac{p}{q}\text{-expansion of some } s \in \mathbb{N}\}$$

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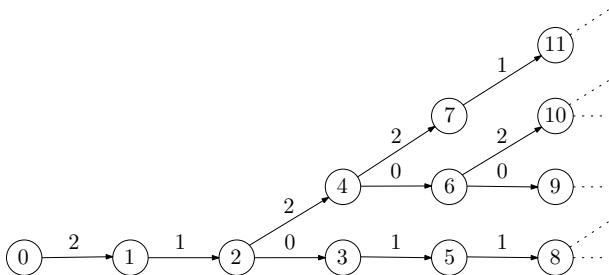
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$$L_{\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid w \text{ is } \frac{p}{q}\text{-expansion of some } s \in \mathbb{N}\}$$

- $L_{\frac{p}{q}}$ is prefix-closed,
- any $u \in \mathcal{A}_p^+$ is a suffix of some $w \in L_{\frac{p}{q}}$,
- $L_{\frac{p}{q}}$ is not context-free (if $q \neq 1$),
- $\pi : \mathcal{A}_p^+ \mapsto \mathbb{Q}$ the evaluation map. If $v, w \in L_{\frac{p}{q}}$, then

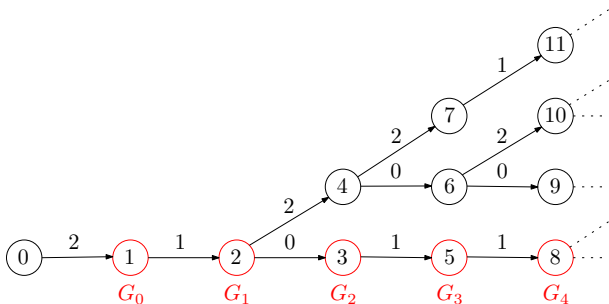
$$v \preceq w \iff \pi(v) \leq \pi(w).$$

$T_{\frac{p}{q}}$ – tree of nonnegative integers



Children of the vertex n are given by $\frac{1}{q}(pn + a) \in \mathbb{N}$, $a \in \mathcal{A}_p$.

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$$G_0 = 1, \quad G_{i+1} = \left\lceil \frac{p}{q} G_i \right\rceil$$

MD algorithm – the negative case

Let s be a negative integer. The $\frac{p}{q}$ -expansion of s is $\langle s \rangle_{\frac{p}{q}} = \cdots a_2 a_1 a_0$ from the MD algorithm:

$$s_0 = s, \quad qs_i = ps_{i+1} + a_i.$$

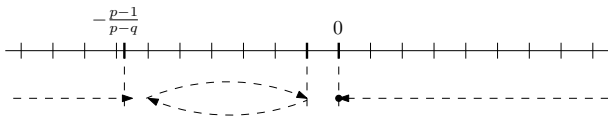
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Properties of $(s_i)_{i \geq 0}$:

- (i) $(s_i)_{i \geq 1}$ is negative,
- (ii) if $s_i < -\frac{p-1}{p-q}$, then $s_i < s_{i+1}$,
- (iii) if $-\frac{p-1}{p-q} \leq s_i < 0$, then $-\frac{p-1}{p-q} \leq s_{i+1} < 0$.



In which fields does it work?

$$s = s_k \left(\frac{p}{q}\right)^k + \sum_{i=1}^{k-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^i$$

We want:

$$\left| s - \sum_{i=1}^{k-1} \frac{a_i}{q} \left(\frac{p}{q}\right)^i \right|_r = |s_k|_r \left| \left(\frac{p}{q}\right)^k \right|_r \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

Hence, if $p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}$, the $\frac{p}{q}$ -expansion of s “works” only in \mathbb{Q}_{r_i} , $i = 1, \dots, k$. The speed of convergence is then $\approx r^{-j_i k}$.

$\frac{p}{q}$ -expansions of negative integers

Proposition

Let k be a positive integer, and denote $B = \left\lfloor \frac{p-1}{p-q} \right\rfloor$. Then:

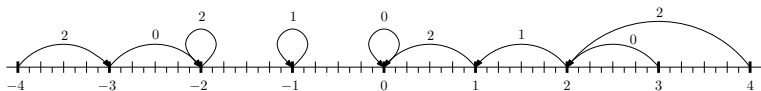
- (i) if $k \leq B$, then $\langle -k \rangle_{\frac{1}{q}\frac{p}{q}} = {}^\omega b$ with $b = k(p-q)$,
- (ii) otherwise, $\langle -k \rangle_{\frac{1}{q}\frac{p}{q}} = {}^\omega bw$ with $w \in \mathcal{A}_p^+$ and $b = B(p-q)$.

$\frac{p}{q}$ -expansions of negative integers

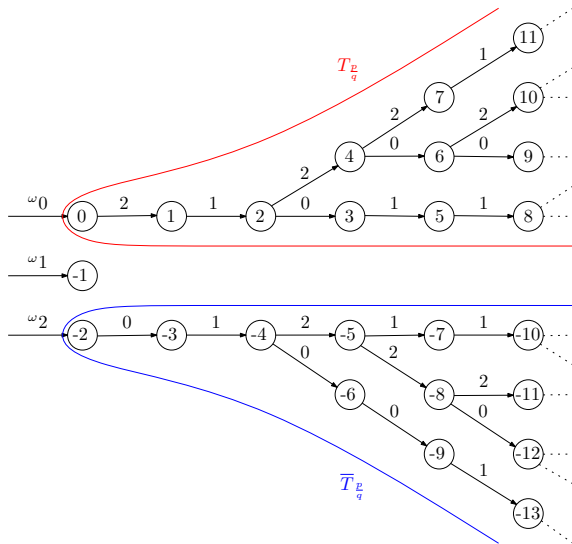
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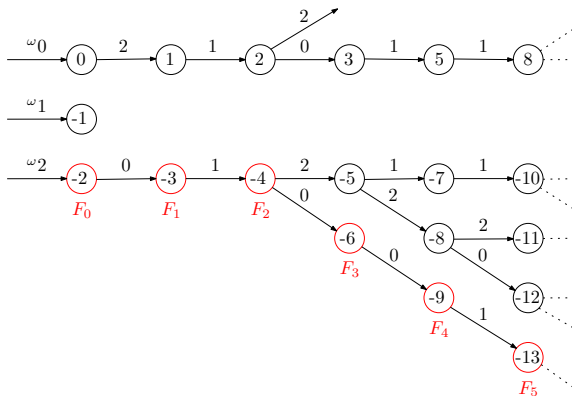
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$\overline{T}_{\frac{p}{q}}$ – tree of negative integers

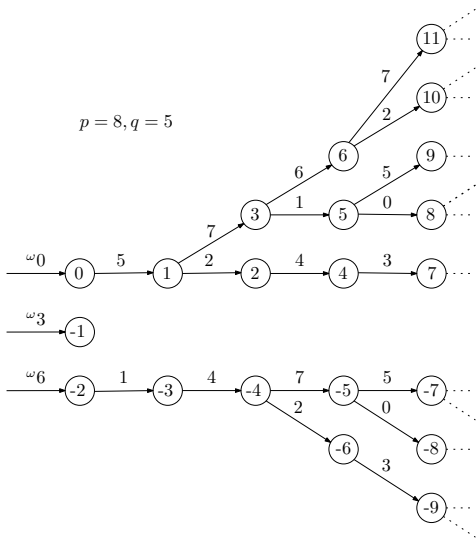


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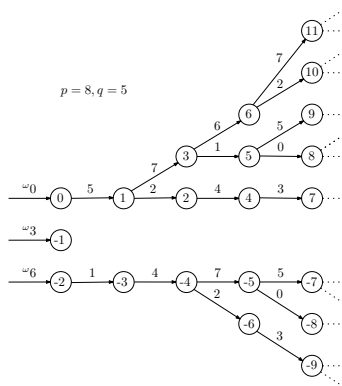


$$F_0 = -B, \quad F_{i+1} = \left\lceil \frac{p}{q} F_i \right\rceil$$

Trees $\overline{T}_{\frac{p}{q}}$ and $T_{\frac{p}{q}}$



Trees $\overline{T}_{\frac{p}{q}}$ and $T_{\frac{p}{q}}$



The trees $\overline{T}_{\frac{p}{q}}$ and $T_{\frac{p}{q}}$ are isomorphic if and only if

$$\frac{p-1}{p-q} \in \mathbb{Z}, \quad \text{i.e., } B(p-q) = p-1$$

$\frac{p}{q}$ -expansion – properties

- for $q = 1$ and p prime we get the standard p -adic representation,

$$\bar{L}_{\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid {}^\omega bw \text{ is } \frac{p}{q}\text{-expansion of } s \leq -B, w_0 \neq b\}$$

- $\bar{L}_{\frac{p}{q}}$ is prefix-closed,
- any $u \in \mathcal{A}_p^+$ is a suffix of some $w \in \bar{L}_{\frac{p}{q}}$,
- $\bar{L}_{\frac{p}{q}}$ is not context-free (if $q \neq 1$).

MD algorithm for rationals

Algorithm (MD algorithm)

Let $x = \frac{s}{t}$, $s, t \in \mathbb{Z}$ co-prime, $s \neq 0$, and $t > 0$ co-prime to p .

Put $s_0 = s$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by

$$q \frac{s_i}{t} = p \frac{s_{i+1}}{t} + a_i.$$

Return the $\frac{p}{q}$ -expansion of x : $\langle x \rangle_{\frac{p}{q}} = \cdots a_2 a_1 a_0$.

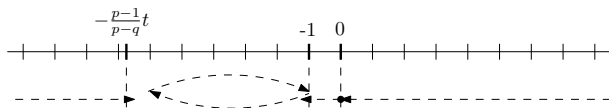
Examples

x	$\langle x \rangle_{\frac{1}{q} \frac{p}{q}}$	$(s_i)_{i \geq 0}$	abs. values
$p = 3, q = 2$			
5	2101	5, 3, 2, 1, 0, 0, ...	all
-5	ω 2012	-5, -3, -2, -2, -2, ...	3
11/4	201	11, 6, 4, 0, 0, ...	all
11/8	ω 1222	11, 2, -4, -8, -8, -8, ...	3
11/5	ω (02)2112	11, 4, 1, -1, -4, -6, -4, -6, ...	3
$p = 30, q = 11$			
5	11 25	5, 1, 0, 0, ...	all
-5	ω 19 8 5	-5, -2, -1, -1, ...	2, 3, 5
11/7	ω (12 21 5) 23 13	11, 1, -5, -3, -6, -5, ...	2, 3, 5

MD algorithm – properties

For the sequence $(s_i)_{i \geq 1}$ from the MD algorithm we have:

- (i) if $s > 0$ and $t = 1$, $(s_i)_{i \geq 1}$ is eventually zero,
- (ii) if $s > 0$ and $t > 1$, $(s_i)_{i \geq 1}$ is either eventually zero or eventually negative,
- (iii) if $s < 0$, $(s_i)_{i \geq 1}$ is negative,
- (iv) if $s_i < -\frac{p-1}{p-q}t$, then $s_i < s_{i+1}$,
- (v) if $-\frac{p-1}{p-q}t \leq s_i < 0$, then $-\frac{p-1}{p-q}t \leq s_{i+1} < 0$.



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- (v) if $-\frac{p-1}{p-q}t \leq s_i < 0$, then $-\frac{p-1}{p-q}t \leq s_{i+1} < 0$.

Proposition

Let $x \in \mathbb{Q}$. Then $\langle x \rangle_{\frac{p}{q}}$ is eventually periodic with period $< \frac{p-1}{p-q}t$.

Finite $\frac{p}{q}$ -expansion

Let x have finite $\frac{p}{q}$ -expansion, then

$$x = \sum_{i=0}^n \frac{a_i}{q} \left(\frac{p}{q}\right)^i = \frac{s}{q^{n+1}}, \quad s > 0.$$

Define for all positive integers m the set

$$\text{INF}(m) := \left\{ s \mid s > 0, \left\langle \frac{s}{q^m} \right\rangle_{\frac{1}{q} \frac{p}{q}} \text{ is infinite} \right\}.$$

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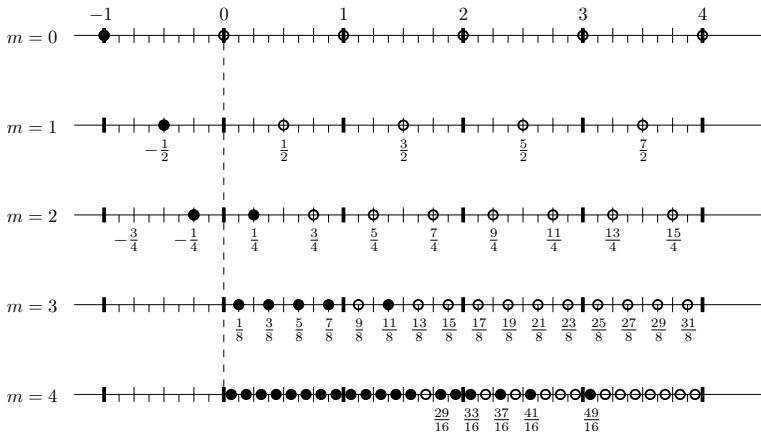
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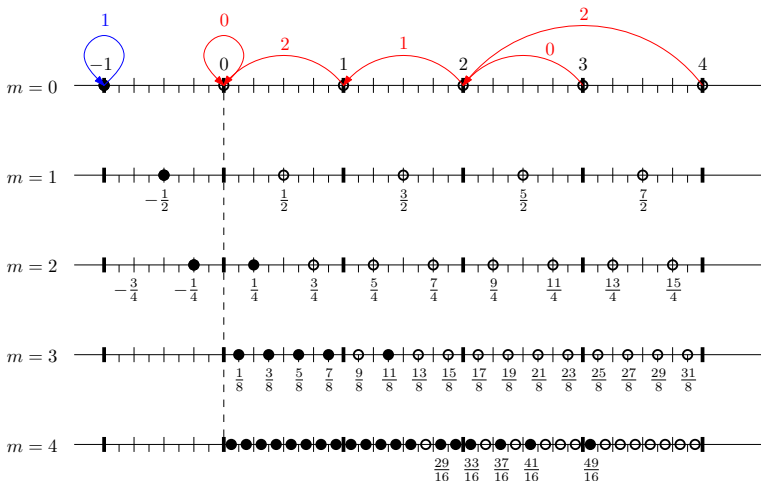
Example ($p = 3, q = 2$)

$$\begin{aligned} \text{INF}(1) &= \emptyset, & \text{INF}(2) &= \{1\}, & \text{INF}(3) &= \{1, 2, 3, 5, 7, 11\}, \\ \text{INF}(4) &= \{1, 2, 3, 4, 5, \dots, 22, 23, 25, 29, 31, 33, 37, 41, 49\}, \\ \#\text{INF}(5) &= 277, & \#\text{INF}(6) &= 1101 \end{aligned}$$

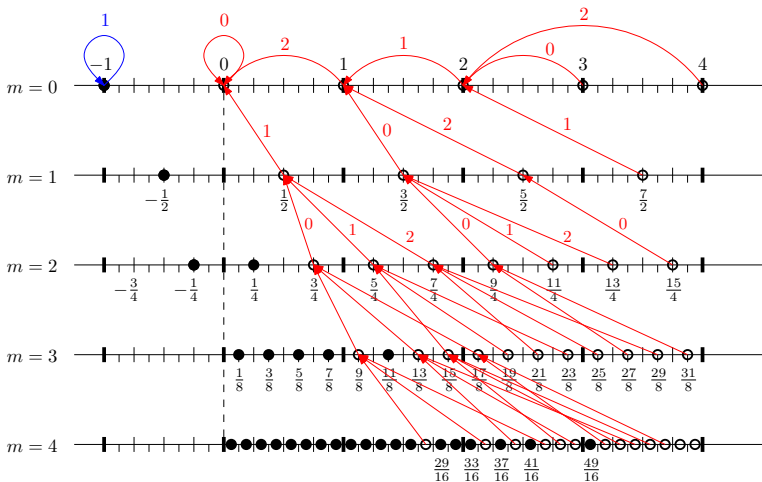
Finite $\frac{p}{q}$ -expansion – II



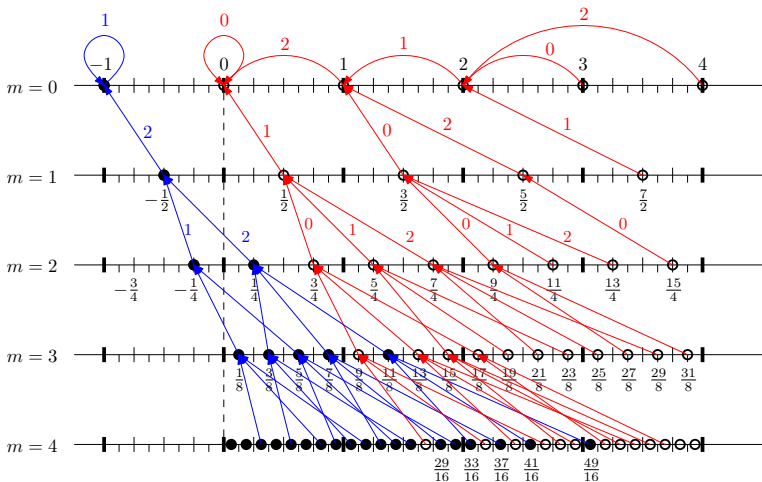
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Finite $\frac{p}{q}$ -expansion – II



Alternative algorithm

Algorithm

Let $x = \frac{s}{q^m}$, s, m positive integers. Put $h_0 = s$ and h_{i+1} and $b_i \in \mathcal{A}_p$ define as follows: for $i = 0, 1, \dots, m-1$ by

$$\frac{h_i}{q^{m-(i+1)}} = p \frac{h_{i+1}}{q^{m-(i+1)}} + b_i,$$

for $i = m, m+1, \dots$ by

$$qh_i = ph_{i+1} + b_i.$$

Return $\mathbf{b} = \dots b_2 b_1 b_0$.

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for $i = m, m+1, \dots$ by

$$qh_i = ph_{i+1} + b_i.$$

Return $\mathbf{b} = \dots b_2 b_1 b_0$.

It holds that $\mathbf{b} = \langle x \rangle_{\frac{p}{q}}$ and

$$s_i = \begin{cases} h_i q^i & i = 0, 1, \dots, m-1 \\ h_i q^m & i = m, m+1, \dots \end{cases}$$

Finite $\frac{p}{q}$ -expansions III

Proposition

$INF(1) = \emptyset$ and $INF(m) = A(m) \cup B(m)$, $m = 2, 3, \dots$, where

$$A(m) = \left\{ -kp + aq^{m-1} \mid k \geq 1, a \in \mathcal{A}_p \right\} \cap \mathbb{N}$$

$$B(m) = \left\{ pk + aq^{m-1} \mid k \in F(m-1), a \in \mathcal{A}_p \right\}.$$

Moreover, for $m \geq 2$ we get

$$\frac{\max INF(m)}{q^m} = \frac{p-1}{p-q} \left(\left(\frac{p}{q} \right)^m - 1 \right) - \left(\frac{p}{q} \right)^m,$$

$\frac{p}{q}$ -expansion of this number is $\omega(p-q)(p-1)^m$.

Finite $\frac{p}{q}$ -expansions – example

Example ($p = 12, q = 5$)

$$INF(1) = \emptyset$$

$$INF(2) = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 13, 14, \dots, 26, 28, 31, 33, 38, 43\}$$

$$\#INF(3) = 1480,$$

$$\#INF(4) = 26100,$$

$$\#INF(5) = 403549.$$

$\frac{p}{q}$ -representation I

Definition

A left infinite word $\cdots a_{-\ell_0+1}a_{-\ell_0}$, $\ell_0 \in \mathbb{N}$, over \mathcal{A}_p is a $\frac{p}{q}$ -representation of $x \in \mathbb{Q}_r$ if $a_{-\ell_0} > 0$ or $\ell_0 = 0$ and

$$x = \sum_{k=-\ell_0}^{\infty} \frac{a_k}{q} \left(\frac{p}{q}\right)^k$$

with respect to $|\cdot|_r$.

If the $\frac{p}{q}$ -representation is not finite (i.e., does not begin in infinitely many zeros), then r is a prime factor of p .

$\frac{p}{q}$ -representation II

Let $p = r_1^{j_1} r_2^{j_2} \cdots r_k^{j_k}$.

Theorem

Let $x \in \mathbb{Q}_{r_i}$ for some $i \in \{1, \dots, k\}$.

- (i) If $k = 1$, there exists a unique $\frac{p}{q}$ -representation of x in \mathbb{Q}_{r_i} .
- (ii) If $k > 1$, there exist uncountably many $\frac{p}{q}$ -representations of x in \mathbb{Q}_{r_i} .

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Let $x \in \mathbb{Q}$ and let $i_1, \dots, i_\ell \in \{1, \dots, k\}$, $\ell < k$, be distinct.

- (i) There exist uncountably many $\frac{p}{q}$ -representations which work in all fields $\mathbb{Q}_{r_{i_1}}, \dots, \mathbb{Q}_{r_{i_\ell}}$.
- (ii) There exists a unique $\frac{p}{q}$ -representation which works in all fields $\mathbb{Q}_{r_1}, \dots, \mathbb{Q}_{r_k}$; namely, the $\frac{p}{q}$ -expansion $\langle x \rangle_{\frac{p}{q}}$.
- (ii) The $\frac{p}{q}$ -expansion $\langle x \rangle_{\frac{p}{q}}$ is the only $\frac{p}{q}$ -representation which is eventually periodic.